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ON THE APPLICABILITY OF DIRECTED FLUID JETS TO NEWTONIAN AND NO--ETC(U)  
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Newtonian and Non-Newtonian Flows

by

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On the Applicability of Directed Fluid Jets to  
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P. M. Naghdi

**Abstract.** This paper elaborates on the applicability of a direct formulation of the theory of jets to Newtonian and non-Newtonian fluid flow problems. Following a general discussion of the nature of direct theories, we record the basic equations of the theory of directed curves for any finite number of directors. Reference is then made to some recent results for incompressible Newtonian viscous flows, including the problem of jet breakup. The major portion of the paper is concerned with application of the direct approach to an incompressible non-Newtonian Poiseuille flow in a circular pipe. The results are compared to those of the three-dimensional theory and are found to include the effect of "normal force" corresponding to the "normal stress" effect.

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## 1. Introduction. General background.

The objective of this paper is to indicate the manner in which a mechanical theory of fluid jets constructed by a direct approach (rather than from the three-dimensional equations) may be used to treat certain one-dimensional Newtonian and non-Newtonian flow problems which may not be tractable with the use of exact three-dimensional equations. This direct approach for fluid jets is based on one-dimensional models called Cosserat (or directed) curves (defined in Sec. 2), which are curves in a Euclidean 3-space endowed with additional structure in the form of kinematical vector fields called directors. Clearly, if full three-dimensional information is desired regarding the motion, deformation and distribution of stresses of the continuum under study, then there is no point in developing a one-dimensional theory. In fact the aim of a one-dimensional theory of the type considered here is to provide only partial information in some specific sense: for example, in the case of a fluid jet or fluid flow in a pipe, information concerning quantities which can be regarded as representing the medium response effectively confined to a neighborhood of a curve as a consequence of the (three-dimensional) motion of the body, or the determination of certain weighted averages of quantities resulting from the (three-dimensional) motion of the body.

The development of the basic theory of Cosserat curves is exact in the sense that it rests on (one-dimensional) postulates valid for nonlinear behavior of materials. By the nature of its construction, the theory necessarily satisfies the requirements of invariance under superposed rigid body motions that arise from physical considerations and, of course, is also consistent and fully invariant in the mathematical sense. Moreover, the development by the direct approach is conceptually simple and is free from the difficulties involved in the approximations usually made in the derivation of jet theories from three-dimensional equations.

It should be remarked here that the use of a direct approach based on Cosserat curves in formulating one-dimensional theories does not mean that one ignores the nature of the field equations in the three-dimensional theory. In fact, some of the developments of the field equations by direct procedure are materially aided or are influenced by available information which can be obtained from the three-dimensional theory. For example, the integrated three-dimensional equations of motion establish guidelines for a statement of the conservation laws of the one-dimensional direct theories, and also provide some insight into the nature of inertia terms and the kinetic energy that appear in the latter theories.\*

Such difficulties as are associated with the derivation of one-dimensional theories from the three-dimensional equations most often arise in the construction of constitutive equations, and it is in fact here that the direct approach offers much appeal. While an approximation to the constitutive equations in the three-dimensional theory retains the constitutive coefficients which have been predetermined within the scope of the three-dimensional theory, the use of such results in a one-dimensional theory may in general lead to incorrect results. In this connection, it should be observed that the constitutive coefficients of the direct theory, in general, may involve contributions from both the material properties of the three-dimensional medium and the local geometry of the body (here, the jet-like body).† The procedure employed in the direct theory, on the other hand, leaves the constitutive coefficients unspecified and while the determination of these coefficients may require substantial effort, they can eventually either be related to those of the exact three-dimensional theory or else be determined by suitable experiments.

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\* See also the remarks following equations (2.15).

† As an illustration, see for example Eqs. (62) of [4] in which the constitutive equations for an incompressible, viscous, elliptical jet depend not only on the shear viscosity but also on the local time-dependent geometry of the cross-section of the jet.

As defined in Appendix A, a jet is a three-dimensional body whose boundary surface has special features and to this extent it is similar to a rod although the nature of the specified surface (or boundary) conditions in the two bodies may be different. Moreover, the kinematics of jets and rods are identical and it is only through the constitutive equations that a distinction appears between rods and jets. In many ways the development of the theory of Cosserat curves is similar to that of the two-dimensional theory of Cosserat (or directed) surfaces which are relevant to fluid sheets and to shells. For a brief historical account of the developments of the theory of directed surfaces and directed curves and for further background information pertaining to a direct formulation of fluid sheets and fluid jets, together with additional references, the reader is referred to a recent expository paper by Naghdi [1]. The first application of the theory of a directed curve to an incompressible Newtonian fluid jet was given by Green and Laws [2] and further work on the subject was contained in a paper by Green [3]. These papers are concerned with a nonlinear theory of circular jets, which includes the effects of both surface tension and gravity. More recent studies on the subject, which will be referred to below, deal with temporal instability and spatial instability of incompressible Newtonian viscous, circular jets [4,5].

The simplest theory appropriate for fluid jets that can be constructed on the basis of a Cosserat curve comprises a material curve and a pair of directors. This theory is suitable for many applications as is clearly evident from the contents of papers of Caulk and Naghdi [4] and of Bogy [5]. The former deals with the onset of breakup of a Newtonian viscous jet, while the latter is mainly concerned with the related problem of jet breakup formulated as a boundary-value problem in connection with ink-jet printing. A brief account of the developments which utilize the theory of a Cosserat curve with two directors is discussed in section 3 of this paper.

Several technologically important problems in rheology, including the swelling effect, are concerned with fluid flow in a pipe. With the hope that it will eventually be possible to treat such problems by direct approach, most of the remainder of the paper is devoted to a Poiseuille flow which requires the use of the next hierarchical theory of Cosserat curves, namely that comprising a material curve with five directors. Instead of developing separately this next hierarchical theory with five directors, it is just as convenient to consider the more general theory of Cosserat curves having any finite number of directors. Thus, in section 2 we construct the theory of directed curves with  $L (\geq 2)$  directors and then briefly discuss the results of the special theories when the number of directors are two and five, respectively. The kinematics of a special Poiseuille flow in a straight circular pipe is considered in section 4 in the context of an approximation procedure whereby the position vector in the three-dimensional theory is taken in the form of a Taylor series expansion and is then assumed to be quadratic in the cross-section coordinates (see Eq. (A18) of Appendix A). The kinematical results of section 4 motivate the choice of the corresponding kinematical ingredients in the theory of a Cosserat curve with five directors. The latter is employed in the discussion of Poiseuille flow in a circular pipe by direct approach in section 5. It should be noted that this solution includes the effect of "normal force" corresponding to the "normal stress" effect in the three-dimensional theory. In fact, the relationship between the two effects is evident from the results of section 6, where the identification of certain quantities in the direct theory is discussed.

## 2. Directed fluid jets with L directors

Deformable media which are modelled by a material curve, embedded in a Euclidean 3-space, together with  $L$  ( $L \geq 2$ ) directors assigned to every point of the curve will be called Cosserat curves or directed curves and may be conveniently referred to as  $\mathcal{R}_K$  ( $K=1,2,\dots$ ). All such directed curves have the same material curve but the number of assigned directors differ for each<sup>†</sup>  $\mathcal{R}_K$ .

Let  $c$ , the material curve of  $\mathcal{R}_K$  in the present configuration at time  $t$ , be defined by its position vector  $\mathbf{r}$  relative to a fixed origin; and let  $\xi$  be a convected (Lagrangian) coordinate identifying points along the curve. Further, let  $L$  directors at  $\mathbf{r}$  be denoted by the vector functions  $\mathbf{d}_{\alpha_1 \alpha_2 \dots \alpha_N}$  ( $\alpha_1, \alpha_2, \dots, \alpha_N = 1, 2, \dots, N$ ;  $N=1, 2, \dots, K$ ), which are assumed to be symmetric in the indices  $\alpha_1 \alpha_2 \dots \alpha_N$ . Then, a motion of the directed curve  $\mathcal{R}_K$  is specified by

$$\mathbf{r} = \hat{\mathbf{r}}(\xi, t), \quad \mathbf{d}_{\alpha_1 \alpha_2 \dots \alpha_N} = \hat{\mathbf{d}}_{\alpha_1 \alpha_2 \dots \alpha_N}(\xi, t), \quad (N=1, 2, \dots, K) \quad (2.1)$$

The velocity and the director velocities are defined by

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{w}_{\alpha_1 \alpha_2 \dots \alpha_N} = \dot{\mathbf{d}}_{\alpha_1 \alpha_2 \dots \alpha_N}, \quad (2.2)$$

where a superposed dot denotes material time differentiation holding  $\xi$  fixed.

Also, the tangent vector to the curve  $c$  denoted by  $\mathbf{a}_3$  is given by

$$\mathbf{a}_3 = \mathbf{a}_3(\xi, t) = \frac{\partial \hat{\mathbf{r}}}{\partial \xi}(\xi, t). \quad (2.3)$$

Corresponding to the requirement in the three-dimensional theory [see (A7) of Appendix A)] that a nonzero volume cannot be continuously deformed into a zero volume, some restrictions on the directors  $\mathbf{d}_{\alpha_1 \alpha_2 \dots \alpha_N}$  are necessary but we leave these unspecified here. It is more convenient to specify such conditions as they arise in special cases of the general theory.

The relationship between the number of directors  $L$  and the number  $K$  in (2.1) which identifies the order of the hierarchical theory of Cosserat

<sup>†</sup>In the absence of the directors, we merely have a one-dimensional curve which can serve as a model for the construction of string theory by direct approach.

curves is seen to be  $L = \sum_{l=1}^K (N+1)$  so that

$$L = \frac{K}{2} (K+1) + K = \frac{K^2 + 3K}{2} . \quad (2.4)$$

This result is in agreement with a corresponding development of the kinematics of a jet-like (or rod-like) body from three-dimensional theory of continuum mechanics contained in the paper of Green et al. [6], Green and Naghdi [7] and Green et al. [8]. According to (2.4), with  $K=1$  the number  $L=2$  and the Cosserat curve  $\mathcal{R}_1 = \mathcal{R}$  consists of a material curve and a pair of directors attached at each point of the curve. Similarly, in the case of the Cosserat curve  $\mathcal{R}_2$ , the number of directors is five and so on.

We assume that the kinetic energy per unit length of the curve  $c$  is given by

$$T = \rho \left[ \frac{1}{2} \tilde{v} \cdot \tilde{v} + \sum_{N=1}^K y^{\alpha_1 \alpha_2 \dots \alpha_N} \tilde{v} \cdot \tilde{w}_{\alpha_1 \alpha_2 \dots \alpha_N} \right. \\ \left. + \frac{1}{2} \sum_{N=1, M=1}^K y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M} \tilde{w}_{\alpha_1 \dots \alpha_N} \cdot \tilde{w}_{\beta_1 \dots \beta_M} \right] \quad (2.5)$$

where  $\rho = \rho(\xi, t)$  is the mass per unit length, the coefficients  $y^{\alpha_1 \dots \alpha_M}$  and  $y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M}$  are functions of  $\xi$  and  $t$ ,  $y^{\alpha_1 \dots \alpha_N}$  are symmetric with respect to indices  $\alpha_1 \dots \alpha_N$ ,  $y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M}$  are symmetric with respect to  $\alpha_1 \dots \alpha_N$  and  $\beta_1 \dots \beta_M$  and are also symmetric with respect to  $\alpha_1 \alpha_2 \dots \alpha_N$  and  $\beta_1 \beta_2 \dots \beta_M$ . In the special case of  $\mathcal{R}_1$  with  $L=2$ , we may use the notations  $w_{\alpha_1} = w_{\alpha}$ ,  $y^{\alpha_1} = y^{\alpha}$ ,  $y^{\alpha_1 \beta_1} = y^{\alpha \beta}$ . (2.6)

We define the momentum corresponding to the velocity  $\tilde{v}$  by

$$\frac{\partial T}{\partial \tilde{v}} = \rho \left[ \tilde{v} + \sum_{N=1}^K y^{\alpha_1 \dots \alpha_N} \tilde{w}_{\alpha_1 \dots \alpha_N} \right] \quad (2.7)$$

per unit length of  $c$ . Similarly, momenta corresponding to director velocities are

$$\frac{\partial T}{\partial \tilde{w}_{\alpha_1 \dots \alpha_N}} = \rho \left[ y^{\alpha_1 \dots \alpha_N} \tilde{v} + \sum_{M=1}^K y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M} \tilde{w}_{\beta_1 \dots \beta_M} \right] \quad (2.8)$$

per unit length of  $c$ .

\* Although the coefficients in (2.5) are regarded as functions of both  $\xi$  and  $t$ , it will be proved presently that they are in fact independent of  $t$ .

Consider now an arbitrary part of the curve  $c$  whose element of arc length is

$$ds = (a_{33})^{\frac{1}{2}} d\xi, \quad a_{33} = \dot{a}_3 \cdot \dot{a}_3. \quad (2.9)$$

Let  $c$  be bounded by  $\xi = \xi_1$  and  $\xi = \xi_2$  ( $\xi_1 < \xi_2$ ) and define the following quantities:

The contact force  $\underline{n} = \underline{n}(\xi, t)$  and the contact director forces  $\underline{p}^{\alpha_1 \alpha_2 \dots \alpha_N} = \underline{p}^{\alpha_1 \alpha_2 \dots \alpha_N}(\xi, t)$ , each a three-dimensional vector field, in the present configuration<sup>†</sup>; the assigned force  $\underline{f} = \underline{f}(\xi, t)$  per unit mass and the assigned director forces  $\underline{\ell}^{\alpha_1 \alpha_2 \dots \alpha_N} = \underline{\ell}^{\alpha_1 \alpha_2 \dots \alpha_N}(\xi, t)$  per unit mass; the intrinsic (curve) director forces  $\underline{\pi}^{\alpha_1 \alpha_2 \dots \alpha_N} = \underline{\pi}^{\alpha_1 \alpha_2 \dots \alpha_N}(\xi, t)$  such that  $(a_{33})^{-\frac{1}{2}} \underline{\pi}^{\alpha_1 \alpha_2 \dots \alpha_N}$  are measured per unit length of  $c$  and make no contributions to the balance laws for moment of momentum and energy; the specific internal energy  $\epsilon = \epsilon(\xi, t)$ ; the specific heat supply  $r = r(\xi, t)$  per unit time; and the heat flux  $h = h(\xi, t)$  along  $c$ , in the direction of increasing  $\xi$ , per unit time. The contact director forces, the intrinsic director forces and the assigned director forces are all assumed to be symmetric with respect to indices  $\alpha_1 \dots \alpha_N$ .

The contact force  $\underline{n}$  has the physical dimension of  $[MT^{-2}]$ , where  $[M]$  and  $[T]$  stand for the physical dimensions of mass and time. The dimensions of the vector fields  $\underline{p}^{\alpha_1 \alpha_2 \dots \alpha_N}$  depend on the choice of the dimensions for the directors.

For example, if each of the vectors  $\underline{d}^{\alpha_1 \alpha_2 \dots \alpha_M}$  is chosen to have the dimension of length, then the coefficients  $y^{\alpha_1 \alpha_2 \dots \alpha_N}$ ,  $y^{\alpha_1 \alpha_2 \dots \alpha_N \beta_1 \beta_2 \dots \beta_M}$  in (2.5) are dimensionless and the vector fields  $\underline{p}^{\alpha_1 \alpha_2 \dots \alpha_N}$  have the same dimensions as  $\underline{n}$ . A parallel remark applies also to the physical dimension of the assigned fields  $\underline{\ell}^{\alpha_1 \alpha_2 \dots \alpha_N}$ . In the present paper we choose the directors to have the dimension of length.

In terms of the foregoing definitions of the various field quantities and with reference to the present configuration at time  $t$ , the conservation laws of

<sup>†</sup> Depending on the physical dimension specified for the directors, the field quantities  $\underline{p}^{\alpha_1 \alpha_2 \dots \alpha_N}$  may also be referred to as the director couples. In fact, for  $M=1$  in (2.1)<sub>2</sub>, corresponding to the Cosserat curve  $\mathcal{R}_1$  the terminology of both the director force vectors and the director couple vectors is used for  $\underline{p}^{\alpha_1} = \underline{p}^{\alpha_1}$  in the literature. A parallel statement holds also for the fields  $\underline{\ell}^{\alpha_1 \alpha_2 \dots \alpha_N}$ .

the purely mechanical theory of a directed curve  $R_K$  are:

$$\frac{d}{dt} \int_{\xi_1}^{\xi_2} \rho ds = 0, \quad \frac{d}{dt} \int_{\xi_1}^{\xi_2} \rho y^{\alpha_1 \dots \alpha_N} ds = 0 \quad (N=1, \dots, K), \quad (2.10)$$

$$\frac{d}{dt} \int_{\xi_1}^{\xi_2} \rho \left[ \tilde{v} + \sum_{N=1}^K y^{\alpha_1 \dots \alpha_N} \tilde{w}_{\alpha_1 \dots \alpha_N} \right] ds = \int_{\xi_1}^{\xi_2} \rho \tilde{f} ds + [\tilde{n}]_{\xi_1}^{\xi_2}, \quad (2.11)$$

$$\begin{aligned} \frac{d}{dt} \int_{\xi_1}^{\xi_2} \rho \left[ y^{\alpha_1 \dots \alpha_N} \tilde{v} + \sum_{M=1}^K y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M} \tilde{w}_{\beta_1 \dots \beta_M} \right] ds \\ = \int_{\xi_1}^{\xi_2} \left[ \rho \ell^{\alpha_1 \dots \alpha_N} - (a_{33})^{-\frac{1}{2}} \tilde{\pi}^{\alpha_1 \dots \alpha_N} \right] ds + [p^{\alpha_1 \dots \alpha_N}]_{\xi_1}^{\xi_2} \quad (N=1, \dots, K), \end{aligned} \quad (2.12)$$

$$\begin{aligned} \frac{d}{dt} \int_{\xi_1}^{\xi_2} \rho \left\{ \tilde{r} \times \tilde{v} + \tilde{r} \times \sum_{N=1}^K y^{\alpha_1 \dots \alpha_N} \tilde{w}_{\alpha_1 \dots \alpha_N} + \sum_{N=1}^K d_{\alpha_1 \dots \alpha_N} \times y^{\alpha_1 \dots \alpha_N} \tilde{v} \right. \\ \left. + \sum_{N=1, M=1}^K d_{\alpha_1 \dots \alpha_N} \times y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M} \tilde{w}_{\beta_1 \dots \beta_M} \right\} ds \\ = \int_{\xi_1}^{\xi_2} \rho \left\{ \tilde{r} \times \tilde{f} + d_{\alpha_1 \dots \alpha_N} \times \ell^{\alpha_1 \dots \alpha_N} \right\} ds \\ + [\tilde{r} \times \tilde{n} + d_{\alpha_1 \dots \alpha_N} \times p^{\alpha_1 \dots \alpha_N}]_{\xi_1}^{\xi_2}, \end{aligned} \quad (2.13)$$

where use is made of the notation

$$[f]_{\xi_1}^{\xi_2} = f(\xi_2) - f(\xi_1),$$

The first two of the above equations, namely  $(2.10)_1$  and  $(2.10)_2$ , represent the mathematical statement of conservation of mass (or inertia) while the remaining equations in the order listed are mathematical statements of the conservation of linear momentum, conservation of director momenta and the conservation of moment of momentum (including contributions from both the ordinary momentum and the director momenta).

We also record here the law of conservation of energy for  $R_K$ , namely

$$\begin{aligned} \frac{d}{dt} \int_{\xi_1}^{\xi_2} [\rho \epsilon + T] ds = \int_{\xi_1}^{\xi_2} \rho \left\{ \tilde{r} + \tilde{f} \cdot \tilde{v} + \sum_{N=1}^K \ell^{\alpha_1 \dots \alpha_N} \cdot \tilde{w}_{\alpha_1 \dots \alpha_N} \right\} ds \\ + [\tilde{n} \cdot \tilde{v} + \sum_{N=1}^K p^{\alpha_1 \dots \alpha_N} \cdot \tilde{w}_{\alpha_1 \dots \alpha_N} - h]_{\xi_1}^{\xi_2}. \end{aligned} \quad (2.14)$$

The assigned field  $\tilde{f}$  in  $(2.10)$  represents the combined effect of (i) an integrated contribution arising from the three-dimensional body force denoted by  $\tilde{f}_b$ , e.g.,

that due to gravitational acceleration, and (ii) an integrated contribution of the stress vector on the lateral surface of the jet-like body<sup>†</sup> denoted by  $\underline{f}_c$ . A parallel statement holds for the assigned fields  $\underline{l}^{\alpha_1 \alpha_2 \dots \alpha_N}$ . Similarly, the assigned heat supply  $r$  represents the combined effect of (i) an integrated contribution arising from the three-dimensional heat supply denoted by  $r_b$ , and (ii) an integrated contribution of the heat supply entering the lateral surface of the jet-like body from the surrounding environment, denoted by  $r_c$ . Thus, we may write

$$\begin{aligned} \underline{f} &= \underline{f}_b + \underline{f}_c, \quad \underline{l}^{\alpha_1 \alpha_2 \dots \alpha_N} = \underline{l}_b^{\alpha_1 \alpha_2 \dots \alpha_N} + \underline{l}_c^{\alpha_1 \alpha_2 \dots \alpha_N}, \\ r &= r_b + r_c. \end{aligned} \quad (2.15)$$

The various quantities in (2.15) are free to be specified in a manner which depends on the particular application in mind and, in the context of the theory of Cosserat curves, the inertia coefficients in (2.4) and the mass density  $\rho$  require constitutive equations. Indeed,  $\underline{f}_c$ ,  $\underline{l}_c^{\alpha_1 \alpha_2 \dots \alpha_N}$  and  $r_c$  (or certain of their features), as well as  $\underline{f}_b$ ,  $\underline{l}_b^{\alpha_1 \alpha_2 \dots \alpha_N}$  and  $r_b$ , can be identified with the corresponding expressions in a derivation from the three-dimensional equations. Likewise, the inertia coefficients in (2.4) and the mass density  $\rho$  may be identified with easily accessible results from the three-dimensional equations.

Assuming smoothness, we may deduce the local form of the balance equations (2.10)<sub>1,2</sub> to be

$$\begin{aligned} \dot{\lambda} &= 0, \quad \lambda^{\alpha_1 \dots \alpha_N} = 0, \quad (N=1, \dots, K), \\ \lambda &= \lambda(\xi) = \rho a_{33}^{\frac{1}{2}}, \end{aligned} \quad (2.16)$$

<sup>†</sup>A definition for jet-like bodies is provided in Appendix A. Its lateral surface is specified by equation (A14).

so that  $\lambda$  and  $y^{\alpha_1 \dots \alpha_N}$  are independent of  $t$  and are functions of  $\xi$  only. Then, the remaining conservation laws (2.11) to (2.13) yield the field equations

$$\frac{\partial \tilde{n}}{\partial \xi} + \lambda \tilde{f} = 0, \quad (2.17)$$

$$\frac{\partial \tilde{p}^{\alpha_1 \dots \alpha_N}}{\partial \xi} + \lambda \tilde{q}^{\alpha_1 \dots \alpha_N} = \tilde{\pi}^{\alpha_1 \dots \alpha_N}, \quad (N=1, \dots, K), \quad (2.18)$$

$$\tilde{a}_3 \times \tilde{n} + \sum_{N=1}^K (\tilde{d}_{\alpha_1 \dots \alpha_N} \times \tilde{\pi}^{\alpha_1 \dots \alpha_N} + \frac{\partial \tilde{d}_{\alpha_1 \dots \alpha_N}}{\partial \xi} \times \tilde{p}^{\alpha_1 \dots \alpha_N}) = 0, \quad (2.19)$$

where

$$\tilde{f} = \tilde{f} - \tilde{v} - \sum_{N=1}^K y^{\alpha_1 \dots \alpha_N} \tilde{w}_{\alpha_1 \dots \alpha_N}, \quad (N=1, \dots, K), \quad (2.20)$$

$$\tilde{q}^{\alpha_1 \dots \alpha_N} = \tilde{\ell}^{\alpha_1 \dots \alpha_N} - y^{\alpha_1 \dots \alpha_N} \tilde{v} - \sum_{M=1}^K y^{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_M} \tilde{w}_{\beta_1 \dots \beta_M}, \quad (N=1, \dots, K). \quad (2.21)$$

With the help of (2.16) to (2.19), the equation for conservation of energy can be reduced to

$$\lambda \dot{x} - \lambda \dot{\epsilon} - \frac{\partial h}{\partial \xi} + P = 0, \quad (2.22)$$

$$P = \tilde{n} \cdot \frac{\partial \tilde{v}}{\partial \xi} + \sum_{N=1}^K \tilde{\pi}^{\alpha_1 \dots \alpha_N} \cdot \tilde{w}_{\alpha_1 \dots \alpha_N} + \sum_{N=1}^K \tilde{p}^{\alpha_1 \dots \alpha_N} \frac{\partial \tilde{w}_{\alpha_1 \dots \alpha_N}}{\partial \xi} = 0, \quad (2.23)$$

where  $P$  is the mechanical power. It may be noted here that within the scope of the purely mechanical theory, the expression (2.23) for mechanical work can be obtained also by considering the rate of work by all contact force and director forces and all assigned force and assigned director forces acting on the curve  $c$  and its end points minus the rate of increase of the kinetic energy and by setting this equal to  $\int_{\xi_1}^{\xi_2} P d\xi$ ; in this connection, see for example [1, Eq. (3.16)].

In the remainder of the paper we shall be concerned with special cases of the above equations appropriate for directed curves  $R_2$  and  $R_1$ . In particular, for the directed curve  $R_2$ , the appropriate differential equations of the

mechanical theory consist of  $(2.16)_2$  with  $N=2$ ,  $(2.16)_{1,3}$ ,  $(2.17)$  and

$$\frac{\partial \tilde{p}^\alpha}{\partial \xi} + \lambda \tilde{q}^\alpha = \tilde{\pi}^\alpha, \quad (2.24)$$

$$\frac{\partial \tilde{p}^{\alpha\beta}}{\partial \xi} + \lambda \tilde{q}^{\alpha\beta} = \tilde{\pi}^{\alpha\beta}, \quad (2.25)$$

$$\tilde{a}_3 \times \tilde{n} + \tilde{d}_\alpha \times \tilde{\pi}^\alpha + \tilde{d}_{\alpha\beta} \times \tilde{\pi}^{\alpha\beta} + \frac{\partial \tilde{d}_\alpha}{\partial \xi} \times \tilde{p}^\alpha + \frac{\partial \tilde{d}_{\alpha\beta}}{\partial \xi} \times \tilde{p}^{\alpha\beta} = 0. \quad (2.26)$$

Also, the expression for the mechanical power in this case is given by

$$P = \tilde{n} \cdot \frac{\partial \tilde{v}}{\partial \xi} + \tilde{\pi}^\alpha \cdot \tilde{w}_\alpha + \tilde{\pi}^{\alpha\beta} \cdot \tilde{w}_{\alpha\beta} + \tilde{p}^\alpha \cdot \frac{\partial \tilde{w}_\alpha}{\partial \xi} + \tilde{p}^{\alpha\beta} \frac{\partial \tilde{w}_{\alpha\beta}}{\partial \xi}. \quad (2.27)$$

A special case of the above general development with two directors, i.e., for a Cosserat curve  $\mathcal{R} = \mathcal{R}_1$ , was first given by Green and Laws [9] in the context of thermomechanics. A related development of a mechanical theory employing three directors is contained in a paper by Cohen [10]. Further, aspects of the basic theory with two directors appropriate for a Cosserat curve  $\mathcal{R}$  are contained in a more recent paper by Green et al. [11].

### 3. Straight Newtonian viscous jets.

We consider here a special case of the general theory of section 2 for a directed curve  $\mathcal{R}_1 = \mathcal{R}$ , which may include the effects of gravity and surface tension, and go on to illustrate its applicability to certain Newtonian flows and related problems of instability or breakup of straight, incompressible, viscous jets of circular cross-section. The motion of the directed curve  $\mathcal{R}$  is specified by  $(2.1)_1$  and by  $(2.1)_2$  with  $N=1$ . Also, the velocity, the director velocity and the inertia coefficients in this case are given by  $(2.2)_1$ ,  $(2.6)_1$  and  $(2.6)_{2,3}$ , respectively. The only kinetical quantities which occur in the theory of a directed curve with only two directors are the forces  $\underset{\sim}{n}, \underset{\sim}{p}^\alpha, \underset{\sim}{\pi}^\alpha$  and the fields  $\underset{\sim}{f}, \underset{\sim}{\ell}^\alpha$ . Thus, in the context of the purely mechanical theory, the local forms of the relevant conservation laws are the mass conservation  $(2.16)_{1,3}$ , the equations of motion  $(2.17)$  and  $(2.24)$  and the consequence of moment of momentum is given by  $(2.26)$  after omitting terms which involve  $\underset{\sim}{d}_{\alpha\beta}$  and  $\partial \underset{\sim}{d}_{\alpha\beta} / \partial \xi$ .

For a constrained theory of the directed curve  $\mathcal{R}$ , we assume that each of the functions  $\underset{\sim}{n}, \underset{\sim}{\pi}^\alpha, \underset{\sim}{p}^\alpha$  is determined to within an additive constraint response so that

$$\underset{\sim}{n} = \bar{\underset{\sim}{n}} + \hat{\underset{\sim}{n}} \quad , \quad \underset{\sim}{\pi}^\alpha = \bar{\underset{\sim}{\pi}}^\alpha + \hat{\underset{\sim}{\pi}}^\alpha \quad , \quad \underset{\sim}{p}^\alpha = \bar{\underset{\sim}{p}}^\alpha + \hat{\underset{\sim}{p}}^\alpha \quad , \quad (3.1)$$

where  $\hat{\underset{\sim}{n}}, \hat{\underset{\sim}{\pi}}^\alpha, \hat{\underset{\sim}{p}}^\alpha$  are determined by constitutive equations and the constraint responses  $\bar{\underset{\sim}{n}}, \bar{\underset{\sim}{\pi}}^\alpha, \bar{\underset{\sim}{p}}^\alpha$  are arbitrary functions of  $\xi, t$  and do no work. For the class of fluid jet problems discussed in [3,4,5], the condition of incompressibility can be shown to yield

$$\frac{d}{dt} [\underset{\sim}{d}_{123}^\alpha] = 0 \quad . \quad (3.2)$$

Then, assuming that for an incompressible Newtonian viscous fluid at constant

\* These are special cases of those defined in section 2 following (2.9).

temperature the constraint response  $\bar{n}, \bar{\pi}^\alpha, \bar{p}^\alpha$  do not depend explicitly on the kinematical quantities

$$\frac{\partial v}{\partial \xi}, \quad w_\alpha, \quad \frac{\partial w}{\partial \xi}, \quad (3.3)$$

it can be shown that [3,12]

$$\bar{n} = -q d_1 \times d_2, \quad \bar{\pi}^\alpha = -q e^{\alpha\beta} d_\beta \times a_3, \quad \bar{p}^\alpha = 0, \quad (3.4)$$

where the Lagrange multiplier  $q$  is an arbitrary scalar function of  $\xi, t$  and  $e^{\alpha\beta}$  is defined by

$$e^{11} = e^{22} = 0, \quad e^{12} = -e^{21} = 1. \quad (3.5)$$

We now specialize the results of the previous section to straight jets of elliptical cross-section. In order to display some details of the kinematics of a straight jet, including the rotation of the directors in a plane normal to the jet axis, it is convenient to introduce a fixed system of rectangular Cartesian coordinates  $(x, y, z)$  with the  $z$ -axis parallel to the jet. Further, let the unit base vectors of the rectangular Cartesian axes be denoted by  $(\underline{i}, \underline{j}, \underline{k})$  and introduce, for later convenience, the additional base vectors

$$\underline{e}_1 = \underline{i} \cos \theta + \underline{j} \sin \theta, \quad \underline{e}_2 = -\underline{i} \sin \theta + \underline{j} \cos \theta, \quad \underline{e}_3 = \underline{k}, \quad (3.6)$$

where  $\theta$  is a smooth function of  $z$  and  $t$ . We assume that the directors are so restricted that they describe an elliptical cross-section of smoothly varying orientation along the length of the jet and that at each  $z = \text{const.}$ , the base vectors  $\underline{e}_1$  and  $\underline{e}_2$  lie along the major and minor axes of the ellipse, respectively. Then, the angle  $\theta = \hat{\theta}(z, t)$ , called the sectional orientation, specifies the orientation of the cross-section at  $z = \text{constant}$  as a function of time. With this background, we now restrict motions of the directed curve  $\mathcal{R}$  such that in

the present configuration at time  $t$ ,

$$\begin{aligned}\tilde{\mathbf{r}} &= \hat{\mathbf{r}}(\xi, t) \quad , \quad \tilde{\mathbf{d}}_1 = \phi_1 \mathbf{e}_1 \quad , \quad \tilde{\mathbf{d}}_2 = \phi_2 \mathbf{e}_2 \quad , \\ \phi_1 &= \hat{\phi}_1(\xi, t) \quad , \quad \phi_2 = \hat{\phi}_2(\xi, t) \quad ,\end{aligned}\tag{3.7}$$

where  $\phi_1$  and  $\phi_2$  measure the semiaxes of the elliptical cross-section. In the case of a circular jet,  $\phi_1 = \phi_2$ .

The complete theory of the directed jet of this section also requires the specification of explicit values for  $\lambda, \mathbf{y}^\alpha, \mathbf{y}^{\alpha\beta}, \mathbf{f}$  and  $\mathbf{l}^\alpha$ . In particular, the values for  $\lambda, \mathbf{y}^\alpha, \mathbf{y}^{\alpha\beta}$  may be obtained by an appeal to certain results from the three-dimensional description of the jet. For this purpose, we may approximate the vector function  $\tilde{\mathbf{r}}^{\wedge*}$  on the right-hand side of (A1) of Appendix A and write this as

$$\mathbf{r}^* = \tilde{\mathbf{r}} + \theta^\alpha \tilde{\mathbf{d}}_\alpha \quad .\tag{3.8}$$

Further, we choose the curve  $\theta^\alpha = 0$  as the line of centroids of the jet-like body and identify this curve with the curve  $c$  in the theory of a Cosserat curve. This leads to the identification

$$\begin{aligned}\lambda &= \rho(a_{33})^{\frac{1}{2}} = \int_Q \rho^* g^{\frac{1}{2}} d\theta^1 d\theta^2 \quad , \\ \lambda \mathbf{y}^\alpha &= \int_Q \rho^* g^{\frac{1}{2}} \theta^\alpha d\theta^1 d\theta^2 = 0 \quad , \quad \lambda \mathbf{y}^{\alpha\beta} = \int_Q \rho^* g^{\frac{1}{2}} \theta^\alpha \theta^\beta d\theta^1 d\theta^2 \quad ,\end{aligned}\tag{3.9}$$

where  $\rho^*$  is the three-dimensional mass density in (A11) and the determinant  $g$  defined by (A5)<sub>3</sub> is calculated from the approximation (3.8). Again, with the use of (3.7) and the equations of motion (2.9)<sub>2,3</sub>, the expressions for  $\tilde{\mathbf{f}}$  and  $\tilde{\mathbf{l}}^\alpha$  can be identified in terms of the integrated body force  $\tilde{\mathbf{f}}^*$  over the cross-section  $Q$  and specified pressure and surface tension over the boundary  $\partial Q$  of  $Q$  (for details, see, for example, Caulk and Naghdi [12]). We observe that since  $\mathbf{y}^\alpha = 0$  by (3.9)<sub>2</sub>, the equations of motion (3.8) and (3.9) assume a

slightly simpler form. We do not record here the system of ordinary differential equations which can be obtained for both inviscid and linear viscous fluids. These are readily available elsewhere and are also discussed in [1].

The development of constitutive equations for an incompressible Newtonian viscous jet of both circular [3] and elliptical [4] cross-sections are readily available in the papers cited, and the constitutive coefficients are identified in terms of the shear viscosity and the geometry of the cross-section of the jet. We do not elaborate here on these results and refer the reader to the papers of Green [3] and Caulk and Naghdi [4].

In the rest of this section we briefly describe some evidence of the relevance and applicability of the direct formulation of viscous fluid jets, especially to problems of instability of viscous jets which utilize the basic equations of the theory of direct fluid jets of this section. For definiteness we consider the linearized version of the relevant equations, neglect the effect of gravity and discuss the onset of breakup of a viscous jet due to surface tension, i.e., the so-called capillary instability. Although our interest here centers mainly on Newtonian viscous jets, in order to assess the nature of the prediction of the direct approach, it is desirable to consider also the breakup of an inviscid jet since this enables us to compare the results with the available exact analysis of the breakup of inviscid jets due to Rayleigh obtained by means of the linearized three-dimensional equations.\* For both inviscid and viscous jets, Rayleigh derived the explicit result that the jet is unstable only in the axisymmetric mode of disturbance. Inasmuch as the direct theory considered here does not begin with the three-dimensional equations, all modes of disturbance which occur in the present one-dimensional

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\*References to Rayleigh's papers on the subject are cited in [4].

direct theory must be examined for stability.\* Thus, in order to allow for the growth of a general disturbance which is not necessarily symmetric, Caulk and Naghdi utilizing the results of [12] derive in [4] a system of linearized equations governing the small motions of a (nonrotating) incompressible inviscid jet of elliptical cross-section superposed on uniform flow of a circular jet. They show that the solution to these linearized equations can be decomposed into two modes, representing a symmetric and an anti-symmetric disturbance in the shape of the free surface. The anti-symmetric mode is stable for all wavelengths, while the symmetric mode is found to be unstable over a range of longer wavelengths. In terms of a description of growth in the unstable mode, comparison of the conclusions is found to agree extremely well with the corresponding exact three-dimensional analysis of Rayleigh.

In the case of a straight incompressible Newtonian viscous jet, through a comparison with available three-dimensional numerical results (Chandrasekhar [13]), the solution obtained is shown to be an improvement over an existing approximate solution of the problem by Weber [14]. A related study by Bogoy [5], concerning the instability of an incompressible viscous liquid jet of circular section, partly overlaps with the work of Caulk and Naghdi [4] on the temporal instability of a viscous jet. Bogoy [5] confines attention to the symmetric mode of disturbance, and considers mainly the spatial instability of a semi-infinite jet formulated as a boundary-value problem. For additional background on breakup and drop formation in viscous jets of circular cross-section, see a recent article by Bogoy [15] which contains additional references on the subject.

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\* If, in the context of the direct theory of this section, the stability analysis is confined through a priori assumptions to the symmetric mode of disturbance only, then any conditions for instability are only sufficient.

#### 4. Kinematics of flow in a pipe: A special Poiseuille flow.

We elaborate here on some three-dimensional kinematical aspects of a fluid flow in a straight circular pipe, using the notation of Appendix A. We first recall that for a Poiseuille flow in a fixed, infinite, circular pipe, the only nonvanishing component of the velocity vector  $\underline{v}^*$  is its axial component and this is a function of the radius of the cross-section. Moreover, the path lines are straight and parallel to the axis of the pipe; and the boundary conditions imposed on the flow are the vanishing of the axial velocity at the wall of the pipe and continuity of the stress tensor at the center line of the pipe. We keep this background information in mind in the development of this section.

Consider now the kinematics of flow in a pipe of uniform circular cross-section in the context of an approximation procedure in which the position vector is approximated by the expression (A18) of Appendix A. Let  $x_i = (x, y, z)$  be a fixed right-handed rectangular Cartesian coordinate system and let the associated unit base vectors be denoted by  $\underline{i}, \underline{j}, \underline{k}$  (see Fig. 2). We identify the center line of the pipe with the  $z$ -axis and specify its cross-section of constant radius  $a$  by

$$x^2 + y^2 = a^2 \quad . \quad (4.1)$$

It is convenient to consider first the kinematics of the flow at time  $\tau \leq t$ . Thus, recalling the notation of Appendix A, let  $\underline{r}^*(\tau) = \underline{r}^*(\theta^i, \tau)$ ,  $\underline{r}(\tau) = \underline{r}(\xi, \tau)$ ,  $\underline{d}_\alpha(\tau) = \underline{d}_\alpha(\xi, \tau)$ ,  $\underline{d}_{\alpha\beta}(\tau) = \underline{d}_{\alpha\beta}(\xi, \tau)$  designate the various quantities in (A18) at time  $\tau$ ; and, with reference to the present configuration at time  $t$ , we adopt the notations  $\underline{r}^* = \underline{r}^*(t)$ ,  $\underline{r} = \underline{r}(t)$ ,  $\underline{d}_\alpha = \underline{d}_\alpha(t)$ ,  $\underline{d}_{\alpha\beta} = \underline{d}_{\alpha\beta}(t)$ . Then, given the approximation (A18) the position vector  $\underline{r}^*(\tau)$  and the velocity  $\underline{v}^*(\tau)$  are:

$$\underline{r}^*(\tau) = \underline{r}(\tau) + \theta^\alpha \underline{d}_\alpha(\tau) + \theta^\alpha \theta^\beta \underline{d}_{\alpha\beta}(\tau) \quad , \quad (4.2)$$

$$\underline{v}^*(\tau) = \underline{v}(\tau) + \theta^\alpha \underline{v}_\alpha(\tau) + \theta^\alpha \theta^\beta \underline{v}_{\alpha\beta}(\tau) \quad , \quad (4.3)$$

where  $\underline{v}(\tau)$ ,  $\underline{w}_\alpha(\tau)$ ,  $\underline{w}_{\alpha\beta}(\tau)$  are defined as in (A19). The position vector of the center line of the pipe and the velocity of a particle on the center line are, respectively,  $\underline{r}^*(0,0,\xi,\tau) = \underline{r}(\xi,\tau)$  and  $\underline{v}^*(0,0,\xi,\tau) = \underline{v}(\tau)$ . Since the z-axis is taken to be coincident with the center line and since in a Poiseuille flow particles on the center line move with a constant velocity, we set

$$\underline{r}(\tau) = \bar{z}(\tau)\underline{k} \quad , \quad \underline{v}(\tau) = \dot{\bar{z}}(\tau)\underline{k} = v\underline{k} \quad , \quad (4.4)$$

where  $\bar{z}(\tau) = \hat{z}(0,0,\xi,\tau)$ , a superposed dot in (4.4) and elsewhere in this section designates material time differentiation with respect to  $\tau$  and  $v$  is a positive constant. At this point, without loss in generality, we identify the convected coordinate  $\xi$  of a particle with its z-coordinate in the reference configuration at time  $\tau_0$ , i.e.,

$$\xi = z(\tau_0) = \hat{z}(\theta^1, \theta^2, \xi, \tau_0) \quad . \quad (4.5)$$

Hence, for particles on the center line of the pipe in the reference configuration we have

$$\begin{aligned} \xi &= \bar{z}(\tau_0) = \hat{z}(0,0,\xi,\tau_0) \quad , \\ \underline{r}(\tau_0) &= \bar{z}(\tau_0)\underline{k} = \xi\underline{k} \quad . \end{aligned} \quad (4.6)$$

We also choose the convected coordinates  $\theta^1$  and  $\theta^2$  such that they are, respectively,  $\frac{1}{a}$  times the x and y coordinates of a particle in the reference configuration at time  $\tau_0$ . Moreover, observing that the x,y coordinates of a particle do not change in the special flow under consideration, we set

$$\begin{aligned} \theta^1 &= \frac{x(\tau_0)}{a} = \frac{x(\tau)}{a} \quad , \quad x(\tau) = \hat{x}(\theta^1, \theta^2, \xi, \tau) \quad , \\ \theta^2 &= \frac{y(\tau_0)}{a} = \frac{y(\tau)}{a} \quad , \quad y(\tau) = \hat{y}(\theta^1, \theta^2, \xi, \tau) \quad . \end{aligned} \quad (4.7)$$

Integration of the second equation in  $(4.4)_2$  and the use of  $(4.6)_1$  results in

$$\bar{z}(\tau) = v(\tau - \tau_0) + \bar{z}(\tau_0) = v(\tau - \tau_0) + \xi , \quad (4.8)$$

so that (4.4)<sub>1</sub> may now be written as

$$\begin{aligned} \underline{r}(\tau) &= \underline{r}(\tau_0) + v(\tau - \tau_0) \underline{k} \\ &= [\xi + v(\tau - \tau_0)] \underline{k} . \end{aligned} \quad (4.9)$$

In order to obtain some restrictions on the functions  $\underline{d}_{\alpha}(\tau)$  and  $\underline{d}_{\alpha\beta}(\tau)$  in the expression for  $\underline{r}^*(\tau)$ , consider first the scalar product of (4.2) and  $\underline{k}$ , i.e.,

$$\begin{aligned} \underline{r}^*(\tau) \cdot \underline{k} &= \bar{z}(\tau) + \theta^1 \underline{d}_{11}(\tau) \cdot \underline{k} + \theta^2 \underline{d}_{22}(\tau) \cdot \underline{k} \\ &+ (\theta^1)^2 \underline{d}_{11}(\tau) \cdot \underline{k} + (\theta^2)^2 \underline{d}_{22}(\tau) \cdot \underline{k} + 2\theta^1 \theta^2 \underline{d}_{12}(\tau) \cdot \underline{k} . \end{aligned} \quad (4.10)$$

In view of the symmetry of the flow, the scalar  $\underline{r}^* \cdot \underline{k}$  must remain unaltered under the transformations

$$(a) \ x \rightarrow -x, y \rightarrow y ; \quad (b) \ x \rightarrow x, y \rightarrow -y ; \quad (c) \ x \rightarrow -x, y \rightarrow -y . \quad (4.11)$$

Hence, from the symmetry transformations (4.11), in the order listed, we obtain

$$\begin{aligned} \theta^1 \underline{d}_{11}(\tau) \cdot \underline{k} + 2\theta^1 \theta^2 \underline{d}_{12}(\tau) \cdot \underline{k} &= 0 , \\ \theta^2 \underline{d}_{22}(\tau) \cdot \underline{k} + 2\theta^1 \theta^2 \underline{d}_{12}(\tau) \cdot \underline{k} &= 0 , \\ \theta^1 \underline{d}_{11}(\tau) \cdot \underline{k} + \theta^2 \underline{d}_{22}(\tau) \cdot \underline{k} &= 0 . \end{aligned} \quad (4.12)$$

It follows from the above three conditions that

$$\underline{d}_{12}(\tau) \cdot \underline{k} = 0 , \quad \underline{d}_{11}(\tau) \cdot \underline{k} = 0 , \quad \underline{d}_{22}(\tau) \cdot \underline{k} = 0 . \quad (4.13)$$

According to (4.13), the vector functions  $\underline{d}_1(\tau)$  and  $\underline{d}_{12}(\tau)$  have no components along the axis of the pipe and thus cannot contribute to the axial velocity  $\underline{v}^*(\tau)$ . Thus, in view of (4.13)<sub>2,3</sub>, we set

$$\underline{d}_1(\tau) = a \underline{i} \quad , \quad \underline{d}_2(\tau) = a \underline{j} \quad . \quad (4.14)$$

Next, using (4.4)<sub>1</sub> and (4.14), from the scalar product of (4.2) and  $\underline{i}$  we have

$$\underline{r}^*(\tau) \cdot \underline{i} = \theta^1 a + (\theta^1)^2 \underline{d}_{11}(\tau) \cdot \underline{i} + (\theta^2)^2 \underline{d}_{22}(\tau) \cdot \underline{i} + 2\theta^1 \theta^2 \underline{d}_{12} \cdot \underline{i} \quad . \quad (4.15)$$

In view of the symmetry of the flow, the scalar  $\underline{r}^* \cdot \underline{i}$  must remain unaltered under the transformation (b) in (4.11) and the resulting expression when compared with (4.15) yields  $4\theta^1 \theta^2 \underline{d}_{12} \cdot \underline{i} = 0$  so that

$$\underline{d}_{12}(\tau) \cdot \underline{i} = 0 \quad . \quad (4.16)$$

Similarly, by requiring that  $\underline{r}^* \cdot \underline{j}$  be unaltered under the transformation (a) in (4.11), we arrive at

$$\underline{d}_{12}(\tau) \cdot \underline{j} = 0 \quad . \quad (4.17)$$

It follows from (4.13)<sub>1</sub>, (4.16) and (4.17) that

$$\underline{d}_{12}(\tau) = \underline{0} \quad . \quad (4.18)$$

On substitution of (4.9), (4.14) and (4.18), (4.2) and (4.3) reduce to

$$\underline{r}^*(\tau) = v(\tau - \tau_0) \underline{k} + \underline{r}(\tau_0) + a(\theta^1 \underline{i} + \theta^2 \underline{j}) + (\theta^1)^2 \underline{d}_{11}(\tau) + (\theta^2)^2 \underline{d}_{22}(\tau) \quad (4.19)$$

and

$$\underline{v}^*(\tau) = v \underline{k} + (\theta^1)^2 \underline{w}_{11} + (\theta^2)^2 \underline{w}_{22} \quad , \quad (4.20)$$

Again the symmetry of the flow requires that at points along the x-axis and the y-axis we have  $\underline{v}^*(0, 1, \xi, \tau) = \underline{v}^*(1, 0, \xi, \tau)$ . This leads us to conclude that

$$\tilde{w}_{11}(\tau) = \tilde{w}_{22}(\tau) \quad (4.21)$$

and (4.20) can be rewritten as

$$\tilde{v}^*(\tau) = v\tilde{k} + [(\theta^1)^2 + (\theta^2)^2] \tilde{w}_{11}(\tau) \quad (4.22)$$

From the condition that the particle velocity vanishes at the wall of the pipe, i.e.,

$$\tilde{v}^*(\tau) = 0 \quad \text{when} \quad (\theta^1)^2 + (\theta^2)^2 = 1 \quad , \quad (4.23)$$

we have

$$\tilde{w}_{11}(\tau) = -v\tilde{k} \quad (4.24)$$

Integration of (4.23) with respect to  $\tau$  yields

$$\tilde{d}_{11}(\tau) = -v(\tau - \tau_0)\tilde{k} \quad , \quad (4.25)$$

where the reference value of  $\tilde{d}_{11}(\tau)$  at  $\tau_0$ , i.e.,  $\tilde{d}_{11}(\tau_0)$ , arising from the integration has been set equal to zero. Recalling (4.20), in a similar manner we also obtain

$$\tilde{d}_{22}(\tau) = \tilde{d}_{11}(\tau) = -v(\tau - \tau_0)\tilde{k} \quad (4.26)$$

We now summarize the foregoing results and record below the expressions for the position vector and the particle velocity both in the reference configuration at time  $\tau_0$  and at time  $\tau$ :

$$\tilde{r}^*(\tau_0) = \xi\tilde{k} + a(\theta^1\tilde{i} + \theta^2\tilde{j}) = \xi\tilde{k} + x\tilde{i} + y\tilde{j} \quad , \quad (4.27)$$

$$\begin{aligned}
\tilde{r}^*(\tau) &= a(\theta^1 \tilde{i} + \theta^2 \tilde{j}) + \{\xi + v(\tau - \tau_0)[1 - (\theta^1)^2 - (\theta^2)^2]\} \tilde{k} \\
&= x \tilde{i} + y \tilde{j} + \{\xi + v(\tau - \tau_0)[1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}]\} \tilde{k} \\
&= r \tilde{e}_r + \{\xi + v(\tau - \tau_0)(1 - \frac{r^2}{a^2})\} \tilde{k} ,
\end{aligned} \tag{4.28}$$

$$\tilde{v}^*(\tau) = [1 - (\theta^1)^2 - (\theta^2)^2] v \tilde{k} = [1 - (\frac{x}{a})^2 - (\frac{y}{a})^2] v \tilde{k} = [1 - (\frac{r}{a})^2] v \tilde{k} , \tag{4.29}$$

where  $r = [x^2 + y^2]^{\frac{1}{2}}$  is the radial distance in cylindrical polar coordinates  $(r, \theta, z)$  with associated orthonormal base vectors  $(\tilde{e}_r, \tilde{e}_\theta, \tilde{k})$ . It is clear from (4.28) that

$$z(\tau) = \xi + v(\tau - \tau_0)(1 - \frac{r^2}{a^2}) , \tag{4.30}$$

which reduces to (4.8) on the center line of the pipe ( $r=0$ ). It is clear from (4.29)<sub>2</sub> that  $(dv^*(\tau)/\partial\tau) = 0$  for fixed values of  $x$  and  $y$  and hence the motion is steady. Also, using (4.29), we may calculate the rate of shear  $\kappa^*$  as ordinarily defined for helical flow in the context of the three-dimensional theory (see, for example, [16, p. 23]). Thus, since the expression (4.29)<sub>3</sub> is a function of  $r$  alone, we have

$$\kappa^* = \left| \frac{\partial v_3^*(\tau)}{\partial r} \right| = \frac{2v}{a} r , \tag{4.31}$$

where  $v_3^*(\tau) = \tilde{v}(\tau) \cdot \tilde{k}$  and the double vertical bar stands for the absolute value.

Figures 2 and 3 depict the position vectors  $\tilde{r}^*(\tau)$  and  $\tilde{r}^*(\tau_0)$  for the special Poiseuille flow in a circular pipe discussed in this section. These figures explicitly indicate the interpretations associated with the functions  $\tilde{d}_\alpha(\tau)$  and  $\tilde{d}_{\alpha\beta}(\tau)$  as given by (4.14), (4.18) and (4.26). It can be seen from Figs. 2(b) and 2(c) that fluid, which in a reference configuration at time  $\tau_0$  occupies a cylindrical region such as that bounded by the normal sections at  $\xi_1$  and  $\xi$ , in the configuration at time  $\tau$  occupies the region between the paraboloids of revolution  $CA'D$  and  $EB'F$ . Figure 3 exhibits how a material line such as  $BE$  in the configuration at time  $\tau_0$  deforms into a portion of a parabola

indicated by B'E.

Before closing this section, we indicate that the flow characterized by (4.28) is isochoric, i.e., the volume of each part of the body remains unchanged. With the use of  $(4.28)_1$ ,  $(A5)_1$  and the notation  $(A2)_3$  of Appendix A, the scalar  $g^{\frac{1}{2}}$  which occurs in (A7) is\*

$$g^{\frac{1}{2}} = \underline{g}_1 \times \underline{g}_2 \cdot \underline{g}_3 = a^2, \quad (4.32)$$

which is a constant. Hence  $\frac{d}{d\tau} g^{\frac{1}{2}} \equiv 0$ .

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\* The expressions for the base vectors  $\underline{g}_i$  are recorded in section 6; see Eqs. (6.5).

5. A Poiseuille flow in the theory of directed jets.

We discuss here a Poiseuille flow for an incompressible viscoelastic fluid in a circular pipe employing the theory of a directed curve  $R_2$  in the absence of the effects of gravity and surface tension. The motion of  $R_2$  is specified by  $(2.1)_1$  and  $(2.1)_2$  with  $K=2$ . It is convenient to consider first the motion of  $R_2$  at time  $\tau \leq t$ . Thus, we write

$$\underline{r}(\tau) = \hat{\underline{r}}(\xi, \tau) \quad , \quad \underline{d}_{\alpha}(\tau) = \hat{\underline{d}}_{\alpha}(\xi, \tau) \quad , \quad \underline{d}_{\alpha\beta}(\tau) = \hat{\underline{d}}_{\alpha\beta}(\xi, \tau) \quad (5.1)$$

and, with reference to the present configuration at time  $t$ , we adopt the notations

$$\underline{r} = \underline{r}(t) \quad , \quad \underline{d}_{\alpha} = \underline{d}_{\alpha}(t) \quad , \quad \underline{d}_{\alpha\beta} = \underline{d}_{\alpha\beta}(t) \quad . \quad (5.2)$$

As in section 4, again let  $x_i = (x, y, z)$  be a fixed right-handed rectangular Cartesian coordinate system but here we use the notation  $\underline{e}_i$  for the unit base vectors in place of  $\underline{i}, \underline{j}, \underline{k}$ .

Let the curve  $c$  of  $R_2$  be the center line of the pipe, and identify the latter with the  $z$ -axis. Also, we choose the directors  $\underline{d}_{\alpha}(\tau)$  such that they describe the symmetries of the flow. In a Poiseuille flow the path lines are straight and parallel to the axis of the pipe, the velocity of the center line is constant and the particles located at the wall of the pipe must remain there. With this background and guided by the kinematical results of section 4, for the motion under consideration we set

$$\begin{aligned} \underline{r}(\tau) &= \underline{r}(\tau_0) + v(\tau - \tau_0)\underline{e}_3 \quad , \quad \underline{r}(\tau_0) = \xi \underline{e}_3 \quad , \quad \underline{d}_{\alpha}(\tau) = a \underline{e}_{\alpha} \quad , \\ \underline{d}_{11}(\tau) &= \underline{d}_{22}(\tau) = -v(\tau - \tau_0)\underline{e}_3 \quad , \quad \underline{d}_{12} = \underline{0} \quad , \end{aligned} \quad (5.3)$$

where  $\underline{r}(\tau_0)$  is the position vector of the fluid particle on the center line in some reference configuration at time  $\tau_0$  and it should be noted that for the

motion (5.3) the tangent vector  $a_3 = e_3$ . By (2.2), the velocity  $\underline{v}$  and the director velocities at time  $\tau$  are

$$\begin{aligned} \underline{v}(\tau) &= v e_3, \quad \underline{w}_{11} = \underline{w}_{22} = -v e_3, \\ \underline{w}_1 &= \underline{w}_2 = \underline{w}_{12} = \underline{0}. \end{aligned} \quad (5.4)$$

With the above choice of kinematical ingredients, it is clear that the motion is steady.

To complete the theory of the directed curve  $R_2$  for an incompressible medium (see the discussion at the end of Appendix A following (A26)), we assume that each of the functions

$$\underline{n}, \underline{\pi}^\alpha, \underline{p}^\alpha, \underline{\pi}^{\alpha\beta}, \underline{p}^{\alpha\beta} \quad (5.5)$$

in (2.27) is determined to within an additive constraint response so that

$$\begin{aligned} \underline{n} &= \bar{\underline{n}} + \hat{\underline{n}}, \quad \underline{\pi}^\alpha = \bar{\underline{\pi}}^\alpha + \hat{\underline{\pi}}^\alpha, \quad \underline{p}^\alpha = \bar{\underline{p}}^\alpha + \hat{\underline{p}}^\alpha, \\ \underline{\pi}^{\alpha\beta} &= \bar{\underline{\pi}}^{\alpha\beta} + \hat{\underline{\pi}}^{\alpha\beta}, \quad \underline{p}^{\alpha\beta} = \bar{\underline{p}}^{\alpha\beta} + \hat{\underline{p}}^{\alpha\beta}, \end{aligned} \quad (5.6)$$

where

$$\hat{\underline{n}}, \hat{\underline{\pi}}^\alpha, \hat{\underline{p}}^\alpha, \hat{\underline{\pi}}^{\alpha\beta}, \hat{\underline{p}}^{\alpha\beta} \quad (5.7)$$

are determined by constitutive equations and the constraint responses

$$\bar{\underline{n}}, \bar{\underline{\pi}}^\alpha, \bar{\underline{p}}^\alpha, \bar{\underline{\pi}}^{\alpha\beta}, \bar{\underline{p}}^{\alpha\beta} \quad (5.8)$$

are arbitrary functions of  $\xi, t$  and do no work. We assume that the constraint responses (5.8) do not depend explicitly on the gradient of the velocities  $\underline{v}(\tau), \underline{w}_\alpha(\tau), \underline{w}_{\alpha\beta}(\tau)$  and since they are workless we have

$$\bar{n} \cdot \underline{v}' + \bar{\pi}^\alpha \cdot \underline{w}_\alpha + \bar{\pi}^{\alpha\beta} \cdot \underline{w}_{\alpha\beta} + \bar{p}^\alpha \cdot \underline{w}'_\alpha + \bar{p}^{\alpha\beta} \cdot \underline{w}'_{\alpha\beta} = 0, \quad (5.9)$$

where for convenience we have introduced the notation

$$(\quad)' \equiv \frac{\partial}{\partial \xi} (\quad). \quad (5.10)$$

Next, we introduce a set of Lagrange multipliers

$$q, q^\gamma, q^{\gamma\eta}, q^{\gamma\eta\sigma}, q^{\gamma\eta\sigma\rho} \quad (5.11)$$

and note that each of the multipliers  $q, q^\gamma = (q^1, q^2), q^{\gamma\eta} = (q^{11}, q^{12}, q^{21}, q^{22}),$  etc., may depend on  $\xi$  and  $\tau$ . Then, incorporating the conditions (A28) to (A31) of Appendix A in (5.9) by the usual procedure, after a lengthy manipulation we obtain the appropriate expressions for the constraint responses (5.8) in terms of (5.11) and the kinematic quantities (5.2) and their spatial derivatives. Since some of the expressions for the constraint responses are quite lengthy and, in any case, will not be needed in the present development, we do not record them here. We note, however, a typical expression, namely

$$\bar{n} = -e^{\alpha\beta} [\frac{1}{2} q d_{\alpha} \times d_{\beta} + q^{\gamma} d_{\alpha} \times d_{\gamma\beta} + 2q^{\gamma\eta} d_{\alpha} \times d_{\gamma\eta\beta}] , \quad q^{\gamma\eta} = q^{\eta\gamma}. \quad (5.12)$$

In view of the choice (5.3), most of the terms in the constraint responses vanish for the special Poiseuille flow under consideration; and, in particular, the expression (5.12) reduces to

$$\bar{n} = -qa^2 \underline{e}_3 - a v (\tau - \tau_0) q^{\gamma} \underline{e}_{\gamma}. \quad (5.13)$$

The response of an incompressible viscoelastic directed fluid jet may depend on the entire history of the functions  $\hat{r}, \hat{d}_{\alpha}, \hat{d}_{\alpha\beta}$  in (5.1) and their derivatives\*. As in the corresponding constrained three-dimensional theory, we assume that the quantities (5.7) are determined by the history of

$$v(\tau) = (a_3(\tau), d_{\alpha}(\tau), d_{\alpha\beta}(\tau), d'_{\alpha}(\tau), d'_{\alpha\beta}(\tau)). \quad (5.14)$$

\* While the derivatives may be calculated from the histories of  $\hat{r}, \hat{d}_{\alpha}, \hat{d}_{\alpha\beta}$  if the latter are sufficiently smooth, their inclusion is for explicitness.

It is clear from (2.3) and (5.3)<sub>2,3,4</sub> that for the flow under discussion the above kinematical quantities reduce to

$$v(\tau) = (e_{\tilde{3}}, ae_{\tilde{\alpha}}, -v(\tau - \tau_0)e_{\tilde{3}}, 0, 0) . \quad (5.15)$$

A typical constitutive equation, for example that for  $\hat{n}_{\tilde{}}$ , may be written as

$$\hat{n}_{\tilde{}} = \int_{-\infty}^t \mathcal{F}(v(\tau)) . \quad (5.16)$$

But, since the argument  $v(\tau)$  of the functional in (5.16) has the form (5.15), it follows that the right-hand side of (5.16) can be regarded as a function of  $v$  and we have

$$\hat{n}_{\tilde{}} = \hat{n}_{\tilde{}}(v) , \quad (5.17)$$

with similar expressions for all other kinematical quantities. We require that all constitutive functionals or constitutive functions, such as those in (5.16) and (5.17), vanish for  $v=0$ . This requirement is similar to the normalization of the constitutive response functionals in the three-dimensional theory.

Referred to the orthonormal basis  $e_{\tilde{i}} = (\tilde{i}, \tilde{j}, \tilde{k})$ , the functions (5.5) can be expressed in terms of their components in the form

$$\begin{aligned} \tilde{r} &= r_{\tilde{i}} e_{\tilde{i}} , \quad \tilde{\pi}^{\alpha} = \pi^{\alpha}_{\tilde{i}} e_{\tilde{i}} , \quad \tilde{p}^{\alpha} = p^{\alpha}_{\tilde{i}} e_{\tilde{i}} , \\ \tilde{\pi}^{\alpha\beta} &= \pi^{\alpha\beta}_{\tilde{i}} e_{\tilde{i}} , \quad \tilde{p}^{\alpha\beta} = p^{\alpha\beta}_{\tilde{i}} e_{\tilde{i}} . \end{aligned} \quad (5.17)$$

Also, since the effect of gravity is neglected,  $\tilde{f}_{\tilde{b}} = 0$ ,  $\tilde{l}_{\tilde{b}}^{\alpha} = 0$ ,  $\tilde{l}_{\tilde{b}}^{\alpha\beta} = 0$  and in the notation of (2.15)<sub>1,2</sub>, we have

$$\tilde{f} = \tilde{f}_{\tilde{c}} , \quad \tilde{l}^{\alpha} = \tilde{l}_{\tilde{c}}^{\alpha} , \quad \tilde{l}^{\alpha\beta} = \tilde{l}_{\tilde{c}}^{\alpha\beta} . \quad (5.18)$$

The quantities (5.18) may be regarded as representing the forces that must be supplied by the pipe wall in order to sustain the assumed flow.

As a consequence of the symmetry of the assumed flow, an examination of the equations of motion suggests that we put

$$\begin{aligned}
n_1 &= n_2 = 0, \\
\pi_{.2}^1 &= \pi_{.3}^1 = \pi_{.1}^2 = \pi_{.3}^2 = 0, \quad p_{.2}^1 = p_{.3}^1 = p_{.1}^2 = p_{.3}^2 = 0, \\
\pi_{.12}^{12} &= \pi_{.21}^{21} = 0, \quad p_{.12}^{12} = p_{.21}^{21} = 0, \\
\pi_{..1}^{11} &= \pi_{..2}^{11} = \pi_{..1}^{22} = \pi_{..2}^{22} = 0, \quad p_{..1}^{11} = p_{..2}^{11} = p_{..1}^{22} = p_{..2}^{22} = 0.
\end{aligned} \tag{5.19}$$

Using the assumptions (5.19), the functions (5.5) can be written in terms of their nonvanishing components as

$$\begin{aligned}
\tilde{n} &= n_3 e_3, \quad \tilde{\pi}^1 = \pi_{.1}^1 e_1, \quad \tilde{\pi}^2 = \pi_{.2}^2 e_2, \quad \pi_{.1}^1 = \pi_{.2}^2, \\
\tilde{p}^1 &= p_{.1}^1 e_1, \quad \tilde{p}^2 = p_{.2}^2 e_2, \quad p_{.1}^1 = p_{.2}^2
\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
\tilde{\pi}^{11} &= \tilde{\pi}^{22} = \pi_{.3}^{11} e_3, \quad \pi_{.3}^{11} = \pi_{.3}^{22}, \\
\tilde{p}^{11} &= \tilde{p}^{22} = p_{.3}^{11} e_3, \quad p_{.3}^{11} = p_{.3}^{22}.
\end{aligned} \tag{5.21}$$

Corresponding to a uniform "pressure" gradient in the fluid, we also assume that  $\partial n_3 / \partial \xi$ , as well as  $\partial p_{.1}^1 / \partial \xi$  are independent of  $\xi$ , i.e.,

$$\frac{\partial n_3}{\partial \xi} = 0, \quad \frac{\partial p_{.1}^1}{\partial \xi} = 0. \tag{5.22}$$

Also, an examination of the equations of motion (2.24) and (2.25), together with (5.19) to (5.21), easily reveals that

$$\begin{aligned}
f_1 &= f_2 = 0, \\
l_{.2}^1 &= l_{.3}^1 = l_{.1}^2 = l_{.3}^2 = 0, \\
l_{..1}^{11} &= l_{..2}^{11} = l_{..1}^{22} = l_{..2}^{22} = 0
\end{aligned} \tag{5.23}$$

so that

$$\begin{aligned}
\tilde{f} &= f_3 e_3, \\
\lambda \tilde{\ell}^1 &= \lambda \ell_{.1}^1 e_1, \quad \lambda \tilde{\ell}^2 = \lambda \ell_{.2}^2 e_2, \quad \ell_{.1}^1 = \ell_{.2}^2, \\
\lambda \tilde{\ell}^{11} &= \lambda \tilde{\ell}^{22} = \lambda \ell_{..3}^{11} e_3, \quad \ell_{..3}^{11} = \ell_{..3}^{22}.
\end{aligned} \tag{5.24}$$

In view of the assumptions (5.18), the constraint responses further simplify. For example, since  $n_1 = n_2 = 0$  by (5.19), then the Lagrange multiplier  $q^V$  in (5.13) must be zero. In this way, we find that the final expression for the constraint responses are given by

$$\begin{aligned}
\tilde{n} &= -q a^2 e_3, \\
\tilde{\pi}^1 &= -q a e_{.1}, \quad \tilde{\pi}^2 = -q a e_{.2}, \\
\tilde{p}^1 &= -2av(\tau - \tau_0) q^{11} e_{.1}, \quad \tilde{p}^2 = -2av(\tau - \tau_0) q^{22} e_{.2}, \quad q^{11} = q^{22}
\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}
\tilde{\pi}^{11} &= \tilde{\pi}^{22} = \tilde{\pi}^{12} = \tilde{\pi}^{21} = 0, \\
\tilde{p}^{11} &= -a^2 q^{11} e_3, \quad \tilde{p}^{22} = -a^2 q^{22} e_3, \\
\tilde{p}^{12} &= \tilde{p}^{21} = 0.
\end{aligned} \tag{5.26}$$

In view of the assumed forms of  $\tilde{n}, \tilde{\pi}^\alpha$  and  $\tilde{\pi}^{\alpha\beta}$  in (5.19) and (5.20), it is at once apparent that the conservation equation (2.26) is identically satisfied. Since the nonvanishing components of the response functions (5.7) have the forms (5.17) and are therefore independent of  $\xi$ , it follows from (2.17), (2.20) and (5.18) that at time  $t$

$$\frac{\partial \tilde{n}_3}{\partial \xi} + \lambda f_3 = 0. \tag{5.27}$$

Recalling (5.20)<sub>1</sub> and (5.25)<sub>1</sub>, from (5.22)<sub>1</sub> we obtain  $\partial q / \partial \xi = 0$ , which upon integration yields

$$q = A_1(t)\xi + A_2(t) \quad (5.28)$$

and hence

$$\bar{n}_3 = -a^2(A_1(t)\xi + A_2(t)) \quad , \quad \lambda f_3 = a^2 A_1(t) \quad , \quad (5.29)$$

where the coefficients  $A_1$  and  $A_2$  are functions of  $t$  only. Consider next the conservation equations (2.24) and recall that  $q^\alpha = \ell^\alpha$  specified by (5.18)<sub>2</sub>. Since  $\hat{\pi}^\alpha$  and  $\hat{p}^\alpha$  are independent of  $\xi$  and since  $\bar{\pi}^\alpha$  may be a function of  $\xi$ , the equation for  $\alpha=1$  in (2.24) after using (5.20) and (5.24)<sub>1</sub> can be written as

$$\frac{\partial p_{.1}^1}{\partial \xi} + \lambda \ell_{.1}^1 = \frac{\Lambda_1}{\pi_{.1}} + \frac{-1}{\pi_{.1}} \quad . \quad (5.30)$$

Now with the use of (5.25)<sub>2,4</sub> and (5.22)<sub>2</sub> and by argument similar to that which resulted in (5.28) and (5.29) we obtain

$$\bar{p}_{.1}^1 = -2av(t-\tau_0)q^{11} = B_1(t)\xi + B_2(t) \quad , \quad (5.31)$$

and

$$\lambda \ell_{.1}^1 = \frac{\Lambda_1}{\pi_{.1}} - B_1(t) - a[A_1(t)\xi - A_2(t)] \quad . \quad (5.32)$$

Since the left-hand side of (5.31) vanishes at  $t=\tau_0$  for all  $\xi$ , it follows that  $B_1(\tau_0) = B_2(\tau_0) = 0$ .

Again, in a similar fashion, from (2.25), (5.23), (5.24), (5.26) and (5.31) we obtain

$$\frac{2v}{a}(t-\tau_0)\bar{p}_{..3}^{11} = B_1(t)\xi + B_2(t) \quad , \quad (5.33)$$

$$\frac{2v}{a}(t-\tau_0)\lambda \ell_{..3}^{11} = \frac{2v}{a}(t-\tau_0)\frac{\Lambda_{11}}{\pi_{..3}} - B_1(t) \quad , \quad \ell_{..3}^{11} = \ell_{..3}^{22} \quad .$$

The expressions (5.29)<sub>1,2</sub>, (5.31)<sub>1</sub>, (5.32) and (5.34) involve the four undetermined functions  $A_1, A_2, B_1, B_2$ . We determine these in the next sections by an appeal to corresponding results in the three-dimensional theory.

Before closing this section, we observe that consistent with  $(5.22)_{1,2}$  and corresponding to uniform "pressure" gradient in the fluid, the difference in the values of  $n_3$  at any two different sections of the pipe is

$$\Delta n_3 = n_3 \Big|_{\xi=\xi_2} - n_3 \Big|_{\xi=\xi_1} = -a^2 A_1 \Delta \xi, \quad (5.35)$$

$$\Delta \xi = \xi_2 - \xi_1$$

while the difference  $\Delta p_{.1}^1$  is given by

$$\Delta p_{.1}^1 = p_{.1}^1 \Big|_{\xi=\xi_2} - p_{.1}^1 \Big|_{\xi=\xi_1} = B_1 \Delta \xi, \quad \Delta p_{.1}^1(\tau_0) = B_1(\tau_0) \Delta \xi = 0. \quad (5.36)$$

6. Determination of the unknown coefficients and the constitutive response functions in the solution of section 5.

This section is concerned with the determination of the unknown coefficients  $A_1, A_2, B_1, B_2$  which occur in (5.29)<sub>1,2</sub> and (5.31) to (5.34), as well as the relationships between the constitutive response functions (5.7) and the corresponding results calculated from the three-dimensional solution of Poiseuille flow. We recall that the final results in section 5 make use of the fact that the gradient of the axial force  $n_3$  along the center line of the pipe is uniform and this corresponds to a uniform "pressure" gradient in the fluid. A result of this kind, if necessary, enables one to identify the coefficients  $A_1$  and  $B_1$  in (5.35) and (5.36) experimentally after an appeal to the expression for the resultants  $\underline{n}$  and  $\underline{p}^\alpha$  in (A20). Here, however, we discuss the determination of the unknown functions  $A_1, A_2, B_1, B_2$  through a comparison of certain expressions in section 5 with corresponding results in the three-dimensional solution of non-Newtonian Poiseuille flow.\*

Preliminary to our main objective, we need to recall certain expressions and results from the three-dimensional theory. Let  $(\underline{e}_r, \underline{e}_\theta, \underline{e}_3)$  denote the unit base vectors of cylindrical polar coordinates  $(r, \theta, z)$  and recall the relationships

$$\begin{aligned}\underline{e}_r &= \cos \theta \underline{e}_1 + \sin \theta \underline{e}_2, & \underline{e}_\theta &= -\sin \theta \underline{e}_1 + \cos \theta \underline{e}_2, \\ \underline{e}_1 &= \cos \theta \underline{e}_r - \sin \theta \underline{e}_\theta, & \underline{e}_2 &= \sin \theta \underline{e}_r + \cos \theta \underline{e}_\theta,\end{aligned}\tag{6.1}$$

where as in section 5 the unit base vectors  $(\underline{e}_1, \underline{e}_2)$  are used in place of  $(\underline{i}, \underline{j})$ . For the axisymmetric problem under discussion, the Cauchy stress tensor referred to  $(\underline{e}_r, \underline{e}_\theta, \underline{e}_3)$  can be written as [see (A9) of Appendix A]

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\* An account of non-Newtonian Poiseuille flow may be found in [16, Sec. 19]. The solution to such viscometric flows within the scope of the theory of viscoelastic fluids was originally given by Rivlin [17].

$$\begin{aligned} \underline{T} = & T_{\langle rr \rangle} \underline{e}_r \otimes \underline{e}_r + T_{\langle 33 \rangle} (\underline{e}_r \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_r) \\ & + T_{\langle \theta\theta \rangle} \underline{e}_\theta \otimes \underline{e}_\theta + T_{\langle 33 \rangle} \underline{e}_3 \otimes \underline{e}_3, \end{aligned} \quad (6.2)$$

where  $T_{\langle rr \rangle}, T_{\langle rz \rangle}, T_{\langle \theta\theta \rangle}, T_{\langle zz \rangle}$  are the nonvanishing physical components of the stress tensor  $\underline{T}$  in cylindrical polar coordinates. For the three-dimensional Poiseuille flow, these stresses are given by [16, p. 20]:

$$\begin{aligned} T_{\langle 33 \rangle} &= -\tau^*(\kappa^*) = -S^*, \quad S^* = \frac{r}{2} f^*, \\ T_{\langle rr \rangle} - T_{\langle \theta\theta \rangle} &= \sigma_1^*(\kappa^*), \\ T_{\langle 33 \rangle} - T_{\langle \theta\theta \rangle} &= \sigma_2^*(\kappa^*), \\ T_{\langle rr \rangle} &= \int_0^r \frac{1}{\zeta} \sigma_1^* \left( \frac{\zeta}{2} f^* \right) d\zeta + z f^* + q^*, \end{aligned} \quad (6.3)$$

where  $\tau^*$  is the shear stress function,  $\sigma_1^*$  and  $\sigma_2^*$  are the normal stress functions, the functions  $\sigma_m^*$  are related to  $\sigma_m^*$ , ( $m=1,2$ ), by

$$\sigma_m^*(\kappa^*) = \sigma_m^*[\tau^{*-1}(S^*)] = \sigma_m^*(S^*), \quad (m=1,2),$$

the rate of shear  $\kappa^*$  is defined by (4.31), the scalar  $q^*$  can be computed once the stresses are known on a cross-section  $z = \text{constant}$  and  $f^*$ , called the specific driving force, is a constant given by

$$\begin{aligned} f^* &= \frac{F^*}{\pi a^2 (z_2 - z_1)}, \\ F^* &= 2\pi \int_0^a \{ T_{\langle 33 \rangle} \Big|_{z_2} - T_{\langle 33 \rangle} \Big|_{z_1} \} r dr. \end{aligned} \quad (6.4)$$

The three-dimensional viscometric functions  $\tau^*, \sigma_1^*, \sigma_2^*$  and the scalars  $q^*, f^*$  correspond, respectively, to  $\tau, \sigma_1, \sigma_2$  and  $q, f$  of [16, Sec. 10]. For further properties of these functions, including the existence of the inverse of  $\tau^*$ , see [16, Sec. 11]. In particular, we note that by virtue of a normalization of the constitutive response functionals, the viscometric functions all vanish

for  $v=0$  (and hence for  $\kappa^*=0$ ).

Using (A5)<sub>1,5</sub> along with (A18) and (4.28), the base vectors  $\underline{g}_i$  and their reciprocals  $\underline{g}^i$  are calculated to be

$$\begin{aligned}\underline{g}_1 &= a\underline{e}_1 - 2v(\tau-\tau_0)\theta^1\underline{e}_3 = a(\cos\theta\underline{e}_r - \sin\theta\underline{e}_\theta) - 2v(\tau-\tau_0)\frac{r}{a}\cos\theta\underline{e}_3, \\ \underline{g}_2 &= a\underline{e}_2 - 2v(\tau-\tau_0)\theta^2\underline{e}_3 = a(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta) - 2v(\tau-\tau_0)\frac{r}{a}\sin\theta\underline{e}_3, \\ \underline{g}_3 &= \underline{e}_3\end{aligned}\quad (6.5)$$

and

$$\begin{aligned}\underline{g}^1 &= \frac{1}{a}\underline{e}_1 = \frac{1}{a}(\cos\theta\underline{e}_r - \sin\theta\underline{e}_\theta), \quad \underline{g}^2 = \frac{1}{a}\underline{e}_2 = \frac{1}{a}(\sin\theta\underline{e}_r + \cos\theta\underline{e}_\theta), \\ \underline{g}^3 &= \frac{2v}{a}(\tau-\tau_0)[\theta^1\underline{e}_1 + \theta^2\underline{e}_2] + \underline{e}_3 = \frac{2v}{a}(\tau-\tau_0)r\underline{e}_r + \underline{e}_3.\end{aligned}\quad (6.6)$$

Then, by (A10)<sub>1,5</sub>, the vectors  $\underline{T}^i$  at time  $t$  may be expressed as

$$\begin{aligned}\frac{1}{a}\underline{T}^1 &= \cos\theta T_{\langle rr \rangle} \underline{e}_r - \sin\theta T_{\langle \theta\theta \rangle} \underline{e}_\theta + \cos\theta T_{\langle r3 \rangle} \underline{e}_3, \\ \frac{1}{a}\underline{T}^2 &= \sin\theta T_{\langle rr \rangle} \underline{e}_r + \cos\theta T_{\langle \theta\theta \rangle} \underline{e}_\theta + \sin\theta T_{\langle r3 \rangle} \underline{e}_3, \\ \underline{T}^3 &= [2v(t-\tau_0)rT_{\langle rr \rangle} + a^2T_{\langle r3 \rangle}] \underline{e}_r + [2v(t-\tau_0)rT_{\langle \theta\theta \rangle} + a^2T_{\langle 33 \rangle}] \underline{e}_3.\end{aligned}\quad (6.7)$$

Clearly with the use of (6.1)<sub>1,2</sub>, the above results may also be expressed relative to the basis  $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ .

Now with the use of (6.7), the definitions (A20) and following a lengthy calculation, the resultant forces at time  $t$  are:

$$\underline{n} = \pi[a^2(f^*\underline{g} + q^*) - \int_0^a \left(\frac{a^2+r^2}{r}\sigma_1^* - 2r\sigma_2^*\right)dr]\underline{e}_3, \quad (6.8)$$

$$\begin{aligned}\underline{n}^1 &= \pi_{.1}^1 \underline{e}_1, \quad \underline{n}^2 = \pi_{.2}^2 \underline{e}_2, \quad \pi_{.1}^1 = \pi_{.2}^2, \\ \pi_{.1}^1 &= \pi a \left[ -\int_0^a \frac{1}{r} \sigma_1^* dr + g f^* + v(t-\tau_0) \frac{f^*}{2} + q^* \right],\end{aligned}\quad (6.9)$$

$$\begin{aligned} \underline{p}^1 &= p_{.1}^1 \underline{e}_1, \quad \underline{p}^2 = p_{.2}^2 \underline{e}_2, \\ p_{.1}^1 = p_{.2}^2 &= \frac{\pi a}{2} \left\{ -f^* \frac{a^2}{4} + v(t-\tau_0) \left[ \frac{1}{4} \int_0^a \frac{r^4 - a^4}{r} \Lambda_1^* dr + \xi f^* + v(t-\tau_0) \left( \frac{f^*}{3} + q^* \right) \right] \right\}, \end{aligned} \quad (6.10)$$

$$\underline{\pi}^{11} = \underline{\pi}^{22} = -\frac{\pi}{4} a^2 f^* \underline{e}_3, \quad \underline{\pi}^{12} = \underline{\pi}^{21} = 0, \quad (6.11)$$

$$\begin{aligned} \underline{p}^{11} = \underline{p}^{22} &= \frac{\pi}{2} \left\{ \int_0^a \left[ -\frac{3r^4 + a^4}{r} \Lambda_1^* + r^3 \Lambda_2^* \right] dr + \frac{a^4}{4} (f^* \xi + q^*) - v(t-\tau_0) f^* \frac{a^4}{12} \right\} \underline{e}_3, \\ \underline{p}^{12} = \underline{p}^{21} &= 0. \end{aligned} \quad (6.12)$$

Also, from (A24), (6.7), the expressions for  $\lambda \underline{f}, \lambda \underline{\ell}^\alpha, \lambda \underline{\ell}^{\alpha\beta}$  at time  $t$  are:

$$\lambda \underline{f} = -\pi a^2 f^* \underline{e}_3, \quad (6.13)$$

$$\lambda \underline{\ell}^1 = \lambda \ell_{.1}^1 \underline{e}_1, \quad \lambda \underline{\ell}^2 = \lambda \ell_{.2}^2 \underline{e}_2, \quad \lambda \ell_{.1}^1 = \lambda \ell_{.2}^2 = \pi a \left[ -\int_0^a \frac{1}{r} \Lambda_1^* dr + \xi f^* + q^* \right], \quad (6.14)$$

and

$$\lambda \underline{\ell}^{11} = \lambda \underline{\ell}^{22} = -\frac{\pi a^2}{2} f^* \underline{e}_3, \quad \lambda \underline{\ell}^{12} = \lambda \underline{\ell}^{21} = 0. \quad (6.15)$$

It is worth observing that while the motion is steady both in the solution via the three-dimensional theory and the direct theory, and while the three-dimensional stress field  $\underline{T}(x,y,z,t)$  is also steady, the expressions for  $\underline{T}^i$  in (6.7) and for the resultants  $\underline{n}, \underline{p}^\alpha$ , etc., obtained from (A20) involve the time  $t$  explicitly. This is due to the fact that the vector  $\underline{T}^i$  in (6.7) and hence the resultants  $\underline{n}, \underline{p}^\alpha$ , etc., are calculated relative to the convected coordinate surface  $\xi = \text{constant}$  which varies with  $z$  and  $t$ , as can readily be seen from Fig. 2 or Eq. (4.30). In this connection, it may be recalled that the base vector  $\underline{g}^3$  which is normal to the surface  $\xi = \text{constant}$  varies with time but is independent of both  $\xi$  and  $z$ .

Now the coefficients  $A_1$  and  $B_1$  in the solution of section 5 may be

determined by requiring that the difference resultants  $\Delta n_3$  and  $\Delta p_{.1}^1$  in (5.35) and (5.36) be the same as the corresponding expressions calculated from (6.8) and (6.10)<sub>3,4</sub>. In this way, we obtain

$$A_1 = -\pi f^* , \quad B_1 = \frac{\pi a}{2} v(t-\tau_0) f^* . \quad (6.16)$$

By (5.29), (6.16)<sub>1</sub> and the fact that  $\hat{n}_3(0) = 0$ , in the solution of section 5, we have  $n_3(0) = \bar{n}_3(0) = -a^2 A_2$ . Comparison of this with the corresponding result calculated from the three-dimensional solution yields

$$A_2 = -\pi q^* . \quad (6.17)$$

It remains to identify the coefficient  $B_2$ . This can be effected with the use of (5.31) and (5.33) and by comparing the results for  $p_{.1}^1$  and  $p_{..3}^{11}$ :

$$B_2 = \frac{\pi a}{2} v(t-\tau_0) q^* . \quad (6.18)$$

An examination of the various results also reveals that the Lagrange multiplier  $q^{11} = q^{22}$  which occurs in (5.22) is not an independent quantity and is related to the multiplier  $q$  in (5.25) by

$$q^{11} = \frac{1}{4} q . \quad (6.19)$$

Having made the identifications (6.16) to (6.18), the constitutive response functions (5.7) may be also identified from the corresponding expressions in the three-dimensional solution. We omit details here but note, in particular, the expression for  $\hat{n}_3$  given by

$$\hat{n}_3 = -\pi \int_0^a \left[ \frac{a^2 + r^2}{r} \sigma_1^* - 2r \sigma_2^* \right] dr , \quad (6.20)$$

which clearly relates the effect of "normal force" to the "normal stress" effect.

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### References

1. P. M. Naghdi, "Fluid jets and fluid sheets: A direct formulation," Proc. 12th Symp. on Naval Hydrodynamics (Wash., D.C., June 1978), to appear.
2. A. E. Green and N. Laws, Int. J. Engng. Sci. 6 (1968) 317.
3. A. E. Green, Int. J. Engng. Sci. 14 (1976) 49.
4. D. A. Caulk and P. M. Naghdi, J. Appl. Mech., to appear = Rept. No. UCB/AM-78-3, Dept. of Mechanical Engineering, University of Calif., Berkeley (July 1978).
5. D. B. Bogy, Phys. Fluids 21 (1978) 190.
6. A. E. Green, N. Laws and P. M. Naghdi, Proc. Cambridge Phil. Soc. 64 (1968) 895.
7. A. E. Green and P. M. Naghdi, Int. J. Solids and Structures 6 (1970) 209.
8. A. E. Green, P. M. Naghdi and M. L. Wenner, Proc. Royal Soc. Lond. A 337 (1974) 451.
9. A. E. Green and N. Laws, Proc. Royal Soc. Lond. A 283 (1966) 145.
10. H. Cohen, Int. J. Engng. Sci. 4 (1966) 511.
11. A. E. Green, P. M. Naghdi and M. L. Wenner, Proc. Royal Soc. Lond. A 337 (1974) 485.
12. D. A. Caulk and P. M. Naghdi, Arch. Rational Mech. Anal., to appear = Rept. No. UCB/AM-77-5, Dept. of Mechanical Engineering, University of Calif., Berkeley (June 1977).
13. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford: Clarendon Press, 1961.
14. C. Weber, ZAMM 11 (1931) 136.
15. D. B. Bogy, Ann. Rev. Fluid Mech. 11 (1979), to appear.
16. B. D. Coleman, H. Markovitz and W. Noll, Viscometric Flow of Non-Newtonian Fluids, Springer-Verlag, New York, 1966.
17. R. S. Rivlin, J. Rational Mech. Anal. 5 (1956) 179.

## Appendix A

We record in this appendix certain details of an approximation procedure [6,7,8] whereby the basic equations governing the motion of jets (or rods) can be derived by integration from the three-dimensional equations of classical continuum mechanics. Although the integrated equations have been obtained previously [6,7,8] for any number of directors, we record here only those which correspond to a directed fluid jet with five directors. We also record the constraint conditions appropriate for the incompressible Poiseuille flow discussed in the present paper.

First we define what is meant by a jet-like body. For this purpose, consider a finite three-dimensional body  $\mathcal{B}$  in a Euclidean 3-space, and let convected coordinates  $\theta^i$  ( $i=1,2,3$ ) be assigned to each particle (or material point) of  $\mathcal{B}$ . Further, let  $\tilde{r}^*$  be the position vector, from a fixed origin, of a typical particle of  $\mathcal{B}$  in the present configuration at time  $t$ . Then, a motion of the (three-dimensional) body is defined by a vector-valued function  $\tilde{r}^{\wedge*}$  which assigns position  $\tilde{r}^*$  to each particle of  $\mathcal{B}$  at each instant of time, i.e.,<sup>§</sup>

$$\tilde{r}^* = \tilde{r}^{\wedge*}(\theta^1, \theta^2, \theta^3, t) . \quad (A1)$$

We assume that the vector function  $\tilde{r}^{\wedge*}$  -- a 1-parameter family of configurations with  $t$  as the real parameter -- is sufficiently smooth in the sense that it is differentiable with respect to  $\theta^i$  and  $t$  as many times as required. It is convenient to set  $\theta^3 = \xi$  and adopt the notation

$$\theta^i = (\theta^\alpha, \xi) , \quad \theta^3 = \xi . \quad (A2)$$

We shall be concerned here with material curves (not necessarily straight lines)

<sup>+</sup>The use of an asterisk attached to various symbols is for later convenience. The corresponding symbols without the asterisks are reserved for different definitions or designations to be introduced later.

<sup>§</sup>Recall that when the particles of a continuum are referred to a convected coordinate system, the numerical values of the coordinates associated with each particle remain the same for all time.

in  $\mathcal{B}$  defined by the equations  $\theta^\alpha = \hat{\theta}^\alpha(\xi)$ , ( $\alpha=1,2$ ); the equation resulting from (1.1) with  $\theta^\alpha = \hat{\theta}^\alpha(\xi)$  represents the parametric form of this material curve in the current configuration and describes a 1-parameter family of curves in space, each of which we assume to be smooth and nonintersecting. We designate the space curve  $\partial^\alpha = 0$  in the current configuration by  $c$ . A point of this curve is specified by the position vector  $\underline{r}$ , relative to the same fixed origin to which  $\underline{r}^*$  is referred, where

$$\underline{r} = \underline{\hat{r}}(\xi, t) = \underline{\hat{r}}^*(0, 0, \xi, t) \quad , \quad (A3)$$

with  $\xi$  belonging to a finite interval  $[\xi_1, \xi_2]$ . Let  $\underline{a}_1, \underline{a}_2$  and  $\underline{a}_3$  denote the unit principal normal, the unit binormal and the tangent vector, respectively, to the curve  $c$ . At each point of  $c$  imagine material filaments lying in the normal plane, i.e., the plane perpendicular to  $\underline{a}_3$ , and forming the normal cross-section<sup>+</sup>  $Q_n$ . The surface swept out by the closed boundary curve  $\partial Q_n$  of  $Q_n$  is called the lateral surface. Such a three-dimensional body is called jet-like if the dimensions in the plane of the normal cross-section are small compared to some characteristic dimension  $L(c)$  of  $c$  (see Fig. 1), e.g., its local radius of curvature  $1/\kappa$ , or the length of  $c$  in the case of a straight curve. A jet-like body is said to be slender if the largest dimension of  $Q_n$  is much smaller than  $L(c)$ . If  $Q_n$  is independent of  $\xi$ , the body is said to be of uniform cross-section, otherwise of variable cross-section. Let the (three-dimensional) jet-like body in some neighborhood of  $c$  be bounded by material surfaces  $\xi = \xi_1$ ,  $\xi = \xi_2$  (indicated in Fig. 1) and a material surface of the form

$$F(\theta^1, \theta^2, \xi) = 0 \quad , \quad (A4)$$

which is chosen such that  $\xi = \text{constant}$  are curved sections of the body bounded

<sup>+</sup>The normal cross-section of a jet is a portion of the normal plane to the curve  $c$ , i.e., the intersection of the body and the normal plane.

by closed curves on this surface with  $c$  lying on or within (A4).<sup>++</sup> In the development of a general theory, it is preferable to leave unspecified the choice of the relation of the curve  $c$  to the boundary surface (A4). In special cases or in specific applications, however, it is necessary to fix the relation of  $c$  to the surface (A4).

We recall the formulae

$$\begin{aligned} \underline{g}_i &= \frac{\partial \mathbf{r}^*}{\partial \theta^i}, \quad g_{ij} = \underline{g}_i \cdot \underline{g}_j, \quad g = \det(g_{ij}), \\ \underline{g}^i \cdot \underline{g}_j &= \delta_j^i, \quad \underline{g}^i = g^{ij} \underline{g}_j, \quad \underline{g}^i \cdot \underline{g}^j = g^{ij}, \end{aligned} \quad (A5)$$

$$dv = g^{\frac{1}{2}} d\theta^1 d\theta^2 d\theta^3 \quad (A6)$$

and further assume that<sup>\*</sup>

$$g^{\frac{1}{2}} = [g_1 g_2 g_3] > 0. \quad (A7)$$

In (A5)-(A6),  $\underline{g}_i$  and  $\underline{g}^i$  are the covariant and the contravariant base vectors at time  $t$ , respectively,  $g_{ij}$  is the metric tensor,  $g^{ij}$  is its conjugate,  $\delta_j^i$  is the Kronecker symbol in 3-space and  $dv$  the volume element in the present configuration.

The velocity vector  $\underline{v}^*$  of a particle of the three-dimensional body in the present configuration is defined by

$$\underline{v}^* = \dot{\underline{r}}^*, \quad (A8)$$

where a superposed dot denotes material time differentiation with respect to  $t$  holding  $\theta^i$  fixed. The stress vector  $\underline{t}$  across a surface in the present configuration with outward unit normal  $\underline{v}^*$  is given by

<sup>++</sup> For most purposes, we could assume a less general form for the lateral bounding surface of the body and write  $F(\theta^1, \theta^2) = 0$  instead of (A4).

<sup>\*</sup> The choice of positive sign in (A7) is for definiteness. Alternatively, for physically possible motions we only need to assume that  $g^{\frac{1}{2}} \neq 0$  with the understanding that in any given motion  $[g_1 g_2 g_3]$  is either  $> 0$  or  $< 0$ . The condition (A7) also requires that  $\theta^i$  be a right-handed coordinate system.

$$\tilde{t} = v_i^* g^{-\frac{1}{2}} \tilde{T}^i = v_i^* \tau^{ik} \tilde{g}_k, \quad \tilde{T} = \tilde{g}_i \otimes g^{-\frac{1}{2}} \tilde{T}^i = \tau^{ik} \tilde{g}_i \otimes \tilde{g}_k, \quad (A9)$$

where

$$\tilde{T}^i = g^{\frac{1}{2}} \tau^{ij} \tilde{g}_j = g^{\frac{1}{2}} \tau_j^i \tilde{g}^j, \quad \tilde{v}^* = v_i^* \tilde{g}^i = v^i \tilde{g}_i, \quad \tau^{ij} = \tilde{g}^i \cdot \tilde{T}^j, \quad (A10)$$

$\tilde{T}$  is the symmetric Cauchy stress tensor and  $\otimes$  denotes the tensor product of two vectors. In terms of quantities defined in (A5)-(A10), the local field equations which follow from the integral forms of the three-dimensional conservation laws for mass, linear momentum and moment of momentum, respectively, are

$$\frac{\dot{\rho}^*}{\rho^* g^{\frac{1}{2}}} = 0, \quad (A11)$$

$$\tilde{T}^i_{,i} + \rho^* \tilde{f}^i g^{\frac{1}{2}} = \rho^* g^{\frac{1}{2}} \dot{\tilde{v}}^i, \quad \tilde{g}_i \times \tilde{T}^i = 0,$$

where  $\rho^*$  is the three-dimensional mass density,  $\tilde{f}^*$  is the body force field per unit mass and a comma denotes partial differentiation with respect to  $\theta^i$ .

Let  $\mathcal{Q}$  (not necessarily the same as the normal cross-section  $\mathcal{Q}_n$  defined above) denote the curved section of the surface  $\xi = \text{constant}$  bounded by  $\partial\mathcal{Q}$ , i.e., a closed curve which is the intersection of the surface  $\xi = \text{constant}$  and the surface (A4). Further, let the points  $\xi_1$  and  $\xi_2$ , with  $\xi_1 < \xi_2$ , form endpoints of a segment of the curve (A3) which we denote by  $\mathcal{P}$ , and designate by  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  the particular sections associated with  $\xi_1$  and  $\xi_2$ , respectively. Now consider an arbitrary part of  $\mathcal{P}^*$  of the three-dimensional region occupied by the body such that: (i)  $\mathcal{P}^*$  contains  $\mathcal{P}$ ; and (ii) the boundary  $\partial\mathcal{P}^*$  of  $\mathcal{P}^*$  consists of the sections  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  and a portion of the surface (A4) bounded at each end by  $\partial\mathcal{Q}_1$  and  $\partial\mathcal{Q}_2$ . A body so described is called a jet-like body and the part  $\mathcal{P}^*$  forms a portion of such a body.

The mass  $m^*$  of a portion of the jet-like body is given by

$$m^*(\mathcal{P}^*) = \int_{\mathcal{P}^*} \rho^* dv = \int_{\mathcal{P}^*} \rho^* g^{\frac{1}{2}} d\theta^1 d\theta^2 d\xi. \quad (A12)$$

where  $\rho^*$  is the mass density of the (three-dimensional) continuum and  $dv$  is the element of volume in the present configuration at time  $t$ . In terms of the segment  $\mathcal{P}$  of the material curve (A3), the same mass has the alternative representation

$$m^* = \int_{\mathcal{P}} \left[ \int_a \rho^* g^{\frac{1}{2}} d\theta^1 d\theta^2 \right] d\xi = \int_{\mathcal{P}} \rho (a_{33})^{\frac{1}{2}} d\xi = \int_{\mathcal{P}} \rho ds = m(\mathcal{P}) , \quad (\text{A13})$$

where the density  $\rho$  per unit length of the curve (A3) is defined by

$$\lambda = \rho (a_{33})^{\frac{1}{2}} = \int_a \rho^* g^{\frac{1}{2}} d\theta^1 d\theta^2 . \quad (\text{A14})$$

In view of (A11)<sub>1</sub>, we note that

$$\dot{\lambda} = 0 . \quad (\text{A15})$$

The curve (A3) is fixed in the jet-like body by the condition [6]

$$\int_a \rho^* g^{\frac{1}{2}} \theta^\alpha d\theta^1 d\theta^2 = 0 . \quad (\text{A16})$$

As in the paper of Green and Naghdi [7], we assume that for the jet-like body described in this appendix, the position vector  $\tilde{r}^*$  in (A1) can be represented by the expansion

$$\tilde{r}^*(\theta^1, \theta^2, \xi, t) = \tilde{r} + \sum_N \theta^{\alpha_1} \theta^{\alpha_2} \dots \theta^{\alpha_N} d_{\alpha_1 \alpha_2 \dots \alpha_N} , \quad (\text{A17})$$

where  $\tilde{r}$  and  $d_{\alpha_1 \alpha_2 \dots \alpha_N}$  are vector functions of  $\xi, t$ . In (A17), the vectors  $d_{\alpha_1 \alpha_2 \dots \alpha_N}$  are completely symmetric in the indices  $\alpha_1, \alpha_2, \dots, \alpha_N$ , the summation is over all values of  $\alpha_1, \alpha_2, \dots, \alpha_N$  and  $N=1, 2, 3, \dots$ . We assume that (A17) may be differentiated as many times as required with respect to any of its variables.

In the rest of this appendix we consider a special case of (A17) and restrict attention to the approximation

$$\tilde{r}^*(\theta^1, \theta^2, \xi, t) = \tilde{r}(\xi, t) + \theta^\alpha d_\alpha(\xi, t) + \theta^\alpha \theta^\beta d_{\alpha\beta}(\xi, t) . \quad (\text{A18})$$

Using this assumption in (A1), (A5) and (A8), we obtain

$$\underline{v}^* = \underline{v} + \theta^2 \underline{w}_\alpha + \theta^\alpha \theta^\beta \underline{w}_{\alpha\beta} , \quad \underline{v} = \dot{\underline{r}} , \quad \underline{w}_\alpha = \dot{\underline{d}}_\alpha , \quad \underline{w}_{\alpha\beta} = \dot{\underline{d}}_{\alpha\beta} , \quad (A19)$$

$$\underline{g}_\alpha = \underline{d}_\alpha + 2\theta^\beta \underline{d}_{\alpha\beta} , \quad \underline{g}_3 = \underline{a}_3 + \theta^\alpha \underline{d}'_\alpha + \theta^\alpha \theta^\beta \underline{d}'_{\alpha\beta} , \quad \underline{a}_3 = \dot{\underline{r}}'$$

where a prime denotes partial differentiation with respect to  $\xi$ . We also recall the following definitions for the resultants  $\underline{n}, \underline{\pi}^\alpha, \underline{p}^\alpha, \underline{\pi}^{\alpha\beta}, \underline{p}^{\alpha\beta}$ :

$$\begin{aligned} \underline{n} &= \int_Q \underline{T}^3 \underline{d}\theta^1 \underline{d}\theta^2 , \quad \underline{\pi}^\alpha = \int_Q \underline{T}^\alpha \underline{d}\theta^1 \underline{d}\theta^2 , \quad \underline{p}^\alpha = \int_Q \underline{T}^3 \theta^\alpha \underline{d}\theta^1 \underline{d}\theta^2 , \\ \underline{\pi}^{\alpha\beta} &= \underline{w}^{\alpha\beta} + \underline{w}^{\beta\alpha} , \quad \underline{w}^{\alpha\beta} = \int_Q \underline{T}^\alpha \theta^\beta \underline{d}\theta^1 \underline{d}\theta^2 , \quad \underline{p}^{\alpha\beta} = \int_Q \underline{T}^3 \theta^\alpha \theta^\beta \underline{d}\theta^1 \underline{d}\theta^2 . \end{aligned} \quad (A20)$$

The equations of motion in terms of the resultants (A20) are obtained by suitable integration of (A11)<sub>2,3</sub> over a section  $Q$  and are given by (for details see [6,7]):

$$\frac{\partial \underline{n}}{\partial \xi} + \lambda \underline{f} = \lambda \dot{\underline{v}} , \quad \frac{\partial \underline{p}^\alpha}{\partial \xi} + \lambda \underline{\ell}^\alpha = \underline{\pi}^\alpha + \lambda \gamma^{\alpha\beta} \dot{\underline{w}}_\beta , \quad (A21)$$

$$\frac{\partial \underline{p}^{\alpha\beta}}{\partial \xi} + \lambda \underline{\ell}^{\alpha\beta} = \underline{\pi}^{\alpha\beta} + \lambda \gamma^{\alpha\beta\gamma\delta} \dot{\underline{w}}_{\gamma\delta} , \quad (A22)$$

$$\underline{a}_3 \times \underline{n} + \underline{d}_\alpha \times \underline{\pi}^\alpha + \underline{d}_{\alpha\beta} \times \underline{\pi}^{\alpha\beta} + \underline{d}'_\alpha \times \underline{p}^\alpha + \underline{d}'_{\alpha\beta} \times \underline{p}^{\alpha\beta} = \underline{0} , \quad (A23)$$

provided that

$$\begin{aligned} \lambda \underline{f} &= \int_Q \rho^* \underline{g}^{\frac{1}{2}} \underline{f}^* \underline{d}\theta^1 \underline{d}\theta^2 + \int_{\partial Q} [(\underline{d}\theta^2 \underline{T}^1 - \underline{d}\theta^1 \underline{T}^2) + \underline{T}^3 (\lambda^2 \underline{d}\theta^1 - \lambda^1 \underline{d}\theta^2)] , \\ \lambda \underline{\ell}^\alpha &= \int_Q \theta^\alpha \rho^* \underline{g}^{\frac{1}{2}} \underline{f}^* \underline{d}\theta^1 \underline{d}\theta^2 + \int_{\partial Q} \theta^\alpha [(\underline{d}\theta^2 \underline{T}^1 - \underline{d}\theta^1 \underline{T}^2) + \underline{T}^3 (\lambda^2 \underline{d}\theta^1 - \lambda^1 \underline{d}\theta^2)] , \\ \lambda \underline{\ell}^{\alpha\beta} &= \int_Q \theta^\alpha \theta^\beta \rho^* \underline{g}^{\frac{1}{2}} \underline{f}^* \underline{d}\theta^1 \underline{d}\theta^2 + \int_{\partial Q} \theta^\alpha \theta^\beta [(\underline{d}\theta^2 \underline{T}^1 - \underline{d}\theta^1 \underline{T}^2) + \underline{T}^3 (\lambda^2 \underline{d}\theta^1 - \lambda^1 \underline{d}\theta^2)] , \end{aligned} \quad (A24)$$

and

$$\lambda \gamma^{\alpha\beta} = \int_Q \rho^* \underline{g}^{\frac{1}{2}} \theta^\alpha \theta^\beta \underline{d}\theta^1 \underline{d}\theta^2 , \quad \lambda \gamma^{\alpha\beta\gamma\delta} = \int_Q \rho^* \underline{g}^{\frac{1}{2}} \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \underline{d}\theta^1 \underline{d}\theta^2 , \quad (A25)$$

where  $\lambda^\alpha = \underline{\lambda} \cdot \underline{g}^\alpha$  and  $\underline{\lambda}$  is a vector tangential to the boundary surface (A4).

If we adopt the approximation (A18) and identify the vectors  $\underline{d}_{\alpha}, \underline{d}_{\beta}$  and the position vector  $\underline{r}$  in (A18) with the directors in  $(2.1)_2$  and the position vector  $(2.1)_1$  of the curve  $c$ , then the development of this appendix and the corresponding results given in section 2 are formally equivalent. In particular, comparison of the equations (A15) and (A21) to (A23) with those in (2.32) reveals a 1-1 correspondence between the two systems of equations provided we identify the expressions (A24) to (A25), respectively, with the assigned fields and the inertia coefficients in the theory of a directed curve with five directors discussed at the end of section 2.

Before closing this appendix, we discuss an appropriate constraint condition arising from incompressibility when the position vector is approximated by (A18). For a three-dimensional incompressible medium, the mass density  $\rho^*$  is constant and by (A11)<sub>1</sub> the condition of incompressibility at time  $\tau \leq t$  is

$$\frac{d}{d\tau} [g_{\alpha\beta\gamma}] = 0, \quad (A26)$$

where  $g^{\frac{1}{2}}$  is defined by (A7)<sub>1</sub>. If we confine attention to the special case in which the position vector is specified by the approximation (A18), then with the use of (A5)<sub>1</sub>, we find

$$\begin{aligned} g^{\frac{1}{2}} = & [d_{\alpha\beta\gamma}] + \theta^\gamma \{ [d_{\alpha\beta\gamma}'] + 2e^{\lambda\gamma} [d_{\alpha\beta\gamma} a_3] \} \\ & + \theta^\lambda \theta^\gamma \{ [d_{\alpha\beta\gamma}'] + 2e^{\mu\gamma} [d_{\alpha\beta\gamma} a_3] \} \\ & + 4 [d_{\alpha\beta\gamma} a_3] + 2\theta^\alpha \theta^\lambda \theta^\gamma \{ 2 [d_{\alpha\beta\gamma} a_3] \} \\ & + e^{\rho\sigma} [d_{\alpha\beta\gamma}'] + 4\theta^\alpha \theta^\beta \theta^\lambda \theta^\gamma [d_{\alpha\beta\gamma} a_3], \end{aligned} \quad (A27)$$

where  $e^{\lambda\gamma}$  and the operator  $( )'$  are respectively defined by (3.5) and (5.10).

After substituting the above result in (A26) and following a routine calculation,

we obtain the following conditions

$$\frac{d}{d\tau} [\underline{a_1 a_2 a_3}] = 0, \quad (A28)$$

$$\frac{d}{d\tau} [\underline{a_1 a_2 a'_3}] + 2e^{\lambda\nu} \frac{\partial}{\partial\tau} [\underline{a_1 a_2 a_3}] = 0, \quad (A29)$$

$$\frac{d}{d\tau} [\underline{a_1 a_2 a'_3}] + e^{\mu\nu} \frac{\partial}{\partial\tau} \{ [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] + 2[\underline{a_1 a_2 a_3}] \} = 0. \quad (A30)$$

$$e^{\rho\nu} \frac{d}{d\tau} \{ [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] \} \\ + [\underline{a_1 a_2 a'_3}] = 0, \quad (A31)$$

$$e^{\rho\sigma} \frac{d}{d\tau} \{ [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] \\ + [\underline{a_1 a_2 a'_3}] + [\underline{a_1 a_2 a'_3}] \\ + [\underline{a_1 a_2 a'_3}] \} = 0. \quad (A32)$$

On the basis of the 1-1 correspondence noted in the preceding paragraph, we may employ the five conditions (A28)-(A32) as constraint conditions for the special motion of the theory of a Cosserat curve of an incompressible fluid discussed in section 5.

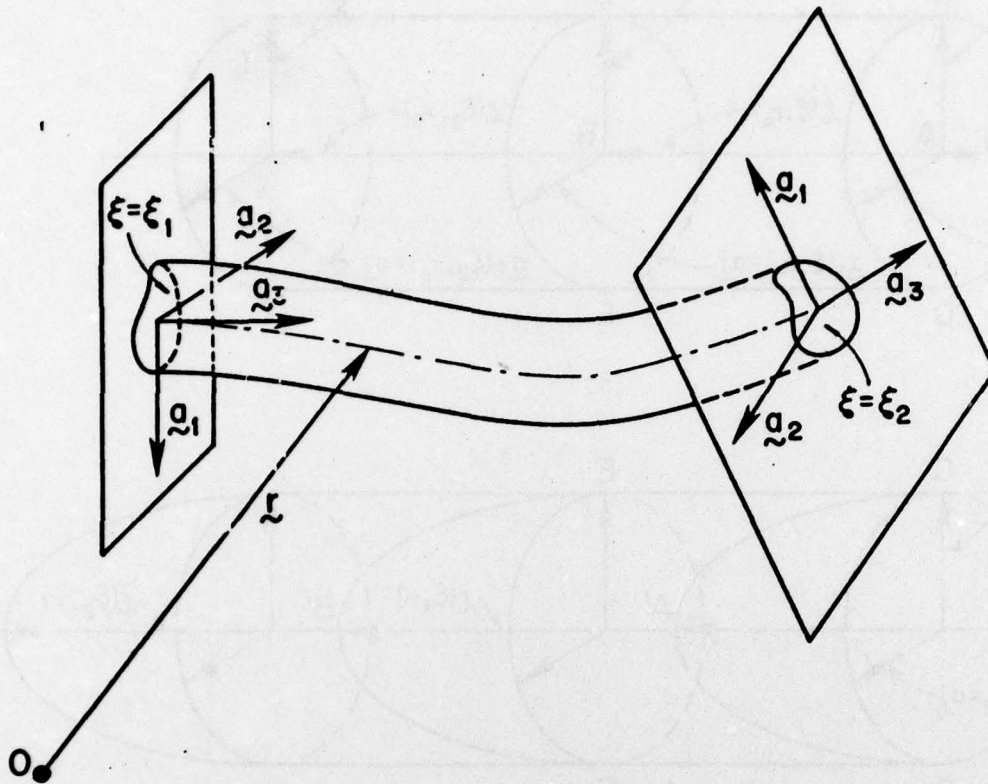


Figure 1: A jet-like body in the present configuration showing the curve  $c$  with position vector  $\underline{r}$  (defined in Appendix A) as the line of centroids and the end normal planes  $\xi = \xi_1$ ,  $\xi = \xi_2$ . Also shown are the unit principal normal  $\underline{a}_1$ , the unit binormal  $\underline{a}_2$  and the tangent vector  $\underline{a}_3$  to the curve  $c$ .

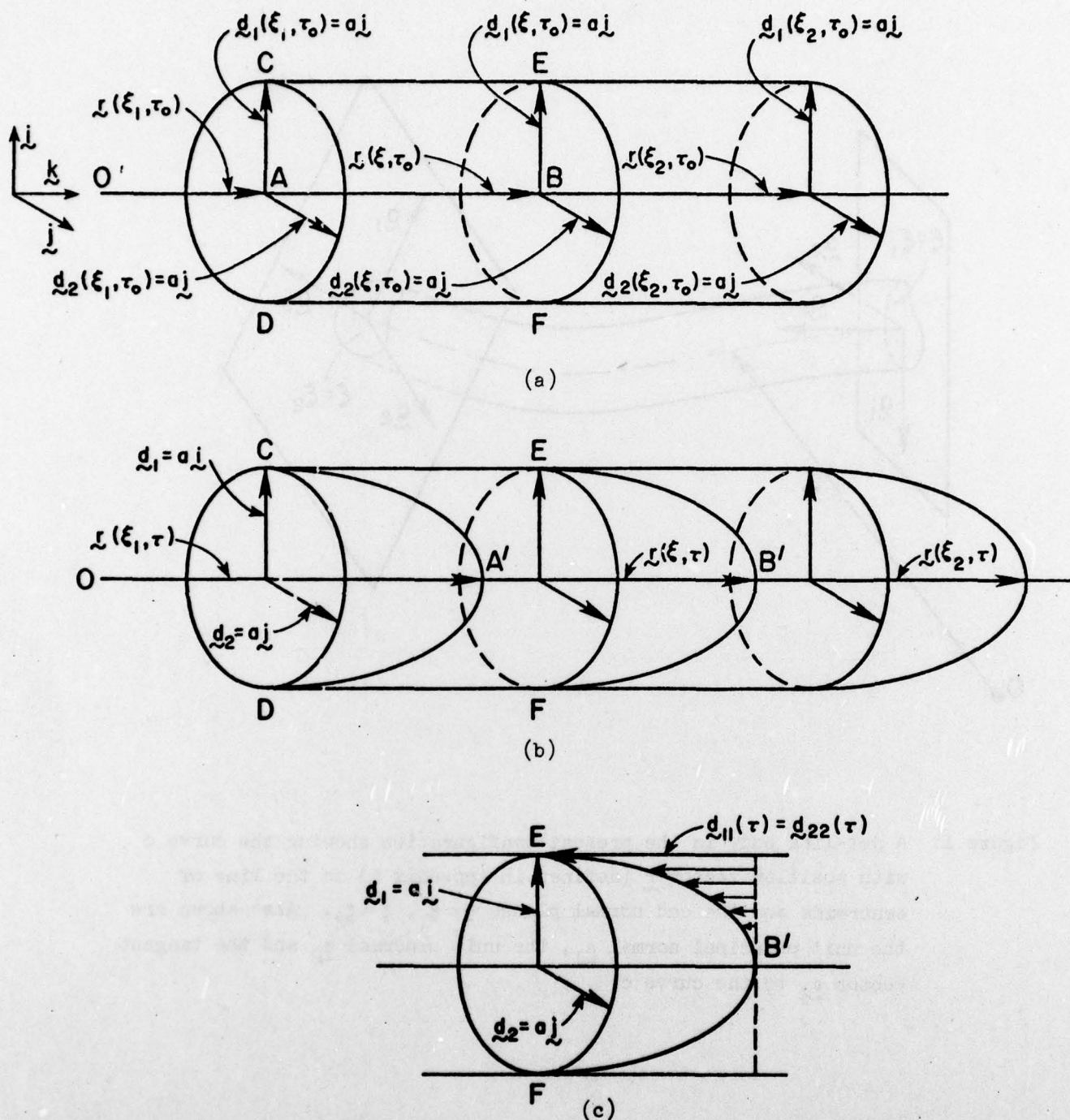


Figure 2. A special Poiseuille flow in a circular pipe: Fig. 2(a) depicts a reference configuration at time  $\tau_0$ ; Fig. 2(b) depicts the corresponding flow configuration at time  $\tau$ ; and Fig. 2(c), along with Fig. 2(b), indicate the locations A' and B' of particles on the center line of the pipe which at time  $\tau_0$  were at A and B, respectively.

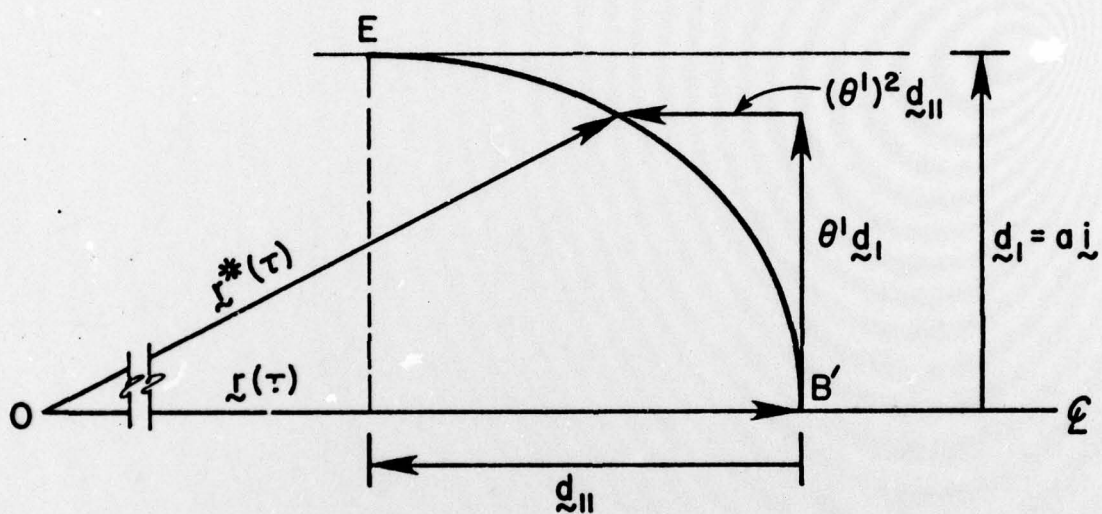


Figure 3: A sketch of an enlarged portion of Fig. 2(c) for the upper-half of a cross-section of the flow in the x-z plane showing the position vector  $\underline{r}^*(\tau)$  of a fluid particle at time  $\tau$ .

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20. (continued)

with application of the direct approach to an incompressible non-Newtonian Poiseuille flow in a circular pipe. The results are compared to those of the three-dimensional theory and are found to include the effect of "normal force" corresponding to the "normal stress" effect.

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